LEIF MEJLBRO

REAL FUNCTIONS OF SEVERAL VARIABLES – SURFACE...
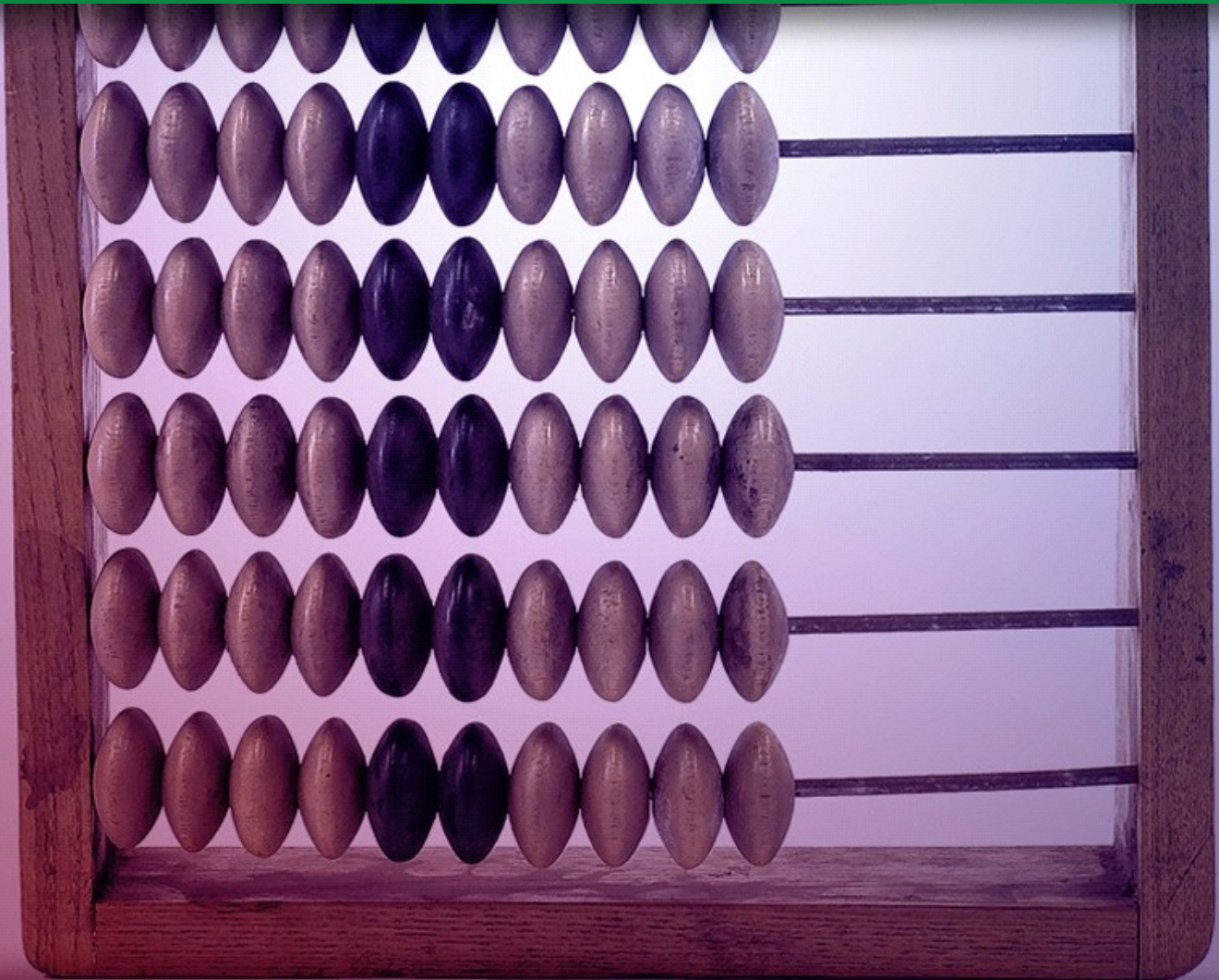
Leif Mejlbro

Real Functions of Several Variables

Examples of Surface Integrals

Calculus 2c-8

Contents

Download free books at BookBooN.com

Preface

In this volume I present some examples of *surface integrals*, cf. also *Calculus 2b, Functions of Several Variables*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I**. *Implementation*. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
15th October 2007

1 Surface integral

Example 1.1 *Calculate in each of the following cases the given surface integral over a surface $\mathcal{F}$, which is the graph of a function in two variables, thus*

$\mathcal{F} = \{(x, y, z) \mid (x, y) \in E,\ z = Z(x, y)\}$.

1) The surface integral $\int_{\mathcal{F}} \sqrt{1 + (x + y + 1)^2}\, dS$, where

$$Z(x, y) = \frac{1}{\sqrt{2}}\, \ln(1 + x + y), \qquad (x, y) \in [0, 1] \times [0, 1].$$

2) The surface integral $\int_{\mathcal{F}} \sqrt{x^2 + y^2}\, dS$, where

$$Z(x, y) = 2 - x^2 - y^2, \qquad \text{for } x^2 + y^2 \le 2.$$

3) The surface integral $\int_{\mathcal{F}} z\, dS$, where

$$Z(x, y) = 2 - x^2 - y^2, \qquad \text{for } x^2 + y^2 \le 2.$$

4) The surface integral $\int_{\mathcal{F}} x^2 \sqrt{1 + x^2 + y^2}\, dS$, where

$$Z(x, y) = xy, \qquad \text{for } x^2 + y^2 \le 1.$$

5) The surface integral $\int_{\mathcal{F}} (a + z)\, dS$, where

$$Z(x, y) = \frac{x^2 - y^2}{a}, \qquad \text{for } x^2 + y^2 \le 2a^2.$$

6) The surface integral $\int_{\mathcal{F}} \dfrac{1}{\sqrt{a^2 + 4x^2 + 4y^2}}\, dS$, where

$$Z(x, y) = \frac{x^2 - y^2}{a}, \qquad \text{for } x^2 + y^2 \le 2a^2.$$

7) The surface integral $\int_{\mathcal{F}} \sqrt{a^2 + 4x^2 + 4y^2}\, dS$, where

$$Z(x, y) = \frac{x^2 - y^2}{a}, \qquad \text{for } x^2 + y^2 \le 2a^2.$$

8) The surface integral $\int_{\mathcal{F}} z^3\, dS$, where

$$Z(x, y) = \sqrt{2a^2 - x^2 - y^2} \qquad \text{for } -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4} \text{ og } 0 \le \varrho \le a\cos(2\varphi).$$
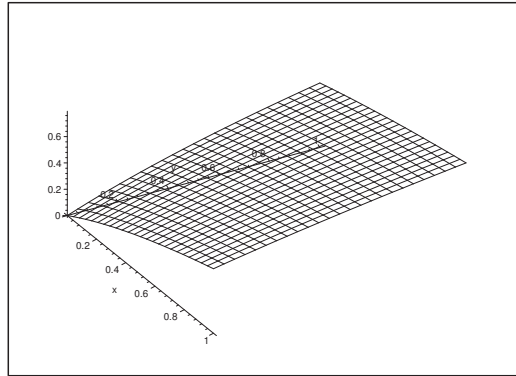
A Surface integrals in rectangular coordinates.

D Find the weight function

$$\|\mathbf{N}\| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} = \sqrt{1 + \|\bigtriangledown g\|^2},$$

and then compute the surface integral.

I 1) We get from $g(x, y) = \dfrac{1}{\sqrt{2}}\, \ln(1 + x + y)$ that

$$\bigtriangledown g = \frac{1}{\sqrt{2}} \cdot \frac{1}{1 + x + y}\, (1, 1),$$

Figure 1: The surface of **Example 1.1.1**.

and as x, $y \geq 0$,

$$\sqrt{1 + \| \nabla f \|^2} = \sqrt{1 + \frac{1}{(1+x+y)^2}} = \frac{\sqrt{1 + (1+x+y)^2}}{1+x+y},$$
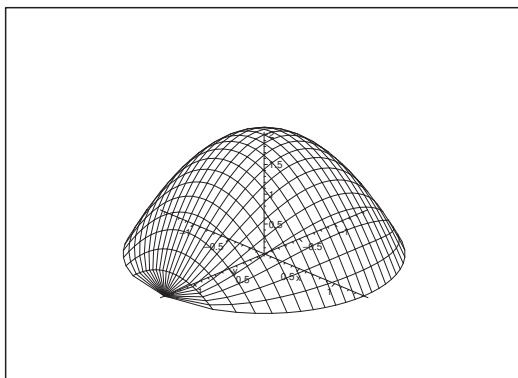
Figure 2: The surface of **Example 1.1.2** and **Example 1.1.3**.

hence

$$\int_{\mathcal{F}} \sqrt{1 + (x+y+1)^2}\, dS = \int_E \frac{1 + (x+y+1)^2}{1+x+y}\, dx\, dy$$

$$= \int_0^1 \left\{ \int_0^1 \left\{ \frac{1}{1+x+y} + x + y + 1 \right\} dy \right\} dx = \int_0^1 \left[\ln(1+x+y) + \frac{1}{2}(x+y+1)^2 \right]_{y=0}^1 dx$$

$$= \int_0^1 \left\{ \ln(x+2) + \frac{1}{2}(x+2)^2 - \ln(x+1) - \frac{1}{2}(x+1)^2 \right\} dx$$

$$= \left[(x+2)\ln(x+2) - (x+1)\ln(x+1) + \frac{1}{6}(x+2)^3 - \frac{1}{6}(x+1)^3 \right]_0^1$$

$$= 3\ln 3 - 2\ln 2 + \frac{1}{6}\cdot 3^3 - \frac{1}{6}\cdot 2^3 - 2\ln 2 - \frac{1}{6}\cdot 2^3 + \frac{1}{6}$$

$$= 3\ln 3 - 4\ln 2 + \frac{1}{6}\{27 - 8 - 8 + 1\} = 3\ln 3 - 4\ln 2 + 2 = \ln\frac{27}{16} + 2.$$

2) We get from $g(x,y) = 2 - x^2 - y^2$ that

$$\bigtriangledown g = (-2x, -2y) = -2(x,y),$$

hence

$$\sqrt{1 + \|\bigtriangledown g\|^2} = \sqrt{1 + 4(x^2 + y^2)}.$$

The method here is that we first transform from the surface $\mathcal{F}$ to the domain of integration E in

rectangular coordinates. Then we continue by transforming the integral into polar coordinates,

$$\int_{\mathcal{F}} \sqrt{x^2 + y^2}\, dS = \int_E \sqrt{x^2 + y^2} \cdot \sqrt{1 + 4(x^2 + y^2)}\, dx\, dy = 2\pi \int_0^{\sqrt{2}} \varrho^2 \sqrt{1 + 4\varrho^2}\, d\varrho$$

$$= 2\pi \int_0^{\mathrm{Arsinh}(2\sqrt{2})} \frac{1}{8}\, \sinh^2 t \cdot \cosh^2 t\, dt = \frac{\pi}{16} \int_0^{\mathrm{Arsinh}(2\sqrt{2})} \sinh^2(2t)\, dt$$

$$= \frac{\pi}{32} \int_0^{\mathrm{Arsinh}(2\sqrt{2})} \{\cosh(4t) - 1\}\, dt = \frac{\pi}{32} \left[\frac{1}{4}\, \sinh(4t) - t \right]_0^{\mathrm{Arsinh}(2\sqrt{2})}$$

$$= \frac{\pi}{32} \left[\frac{1}{2}\, \sinh(2t) \cosh(2t) \right]_0^{\mathrm{Arsinh}(2\sqrt{2})} - \frac{\pi}{32} \ln\left(2\sqrt{2} + \sqrt{1 + (2\sqrt{2})^2} \right)$$

$$= \frac{\pi}{32} \left[\sinh t \cdot \cosh t(1 + 2\sinh^2 t) \right]_0^{\mathrm{Arsinh}(2\sqrt{2})} - \frac{\pi}{32} \ln(3 + 2\sqrt{2})$$

$$= \frac{\pi}{32} \cdot 2\sqrt{2} \cdot \sqrt{1 + (2\sqrt{2})^2} \cdot \left\{ 1 + 2 \cdot (2\sqrt{2})^2 \right\} - \frac{\pi}{32} \ln\left\{ (1 + \sqrt{2})^2 \right\}$$

$$= \frac{\pi}{32} \cdot 2\sqrt{2} \cdot 3 \cdot (1 + 2 \cdot 8) - \frac{\pi}{16} \ln(1 + \sqrt{2}) = \frac{\pi}{16} (51\sqrt{2} - \ln(1 + \sqrt{2})).$$

3) We shall here integrate over the same surface as in **Example 1.1.2**. We can therefore reuse the previous result

$$\sqrt{1 + \| \bigtriangledown g \|^2} = \sqrt{1 + 4(x^2 + y^2)}.$$

If we put $t = 4\varrho^2 + t$, we get the surface integral

$$\int_{\mathcal{F}} z\, dS = \int_E (2 - x^2 - y^2)\sqrt{1 + 4(x^2 + y^2)}\, dx\, dy = 2\pi \int_0^{\sqrt{2}} (2 - \varrho^2)\sqrt{1 + 4\varrho^2} \cdot \varrho\, d\varrho$$

$$= \frac{2\pi}{8} \int_1^9 \left\{ 2 - \frac{1}{4}(t - 1) \right\} \sqrt{t}\, dt = \frac{\pi}{4} \int_1^9 \left\{ \frac{9}{4}\, t^{\frac{1}{2}} - \frac{1}{4}\, t^{\frac{3}{2}} \right\} dt = \frac{\pi}{16} \left[9 \cdot \frac{2}{3}\, t^{\frac{3}{2}} - \frac{2}{5}\, t^{\frac{5}{2}} \right]_1^9$$

$$= \frac{\pi}{8} \left[3 t^{\frac{3}{2}} - \frac{1}{5}\, t^{\frac{5}{2}} \right]_1^9 = \frac{\pi}{8} \left\{ 3 \cdot 27 \cdot \frac{1}{5} \cdot 243 - 3 + \frac{1}{5} \right\} = \frac{\pi}{8} \left(78 - \frac{242}{5} \right)$$

$$= \frac{\pi}{4} \left\{ 39 - \frac{121}{5} \right\} = \frac{\pi}{20}(195 - 121) = \frac{\pi}{20} \cdot 74 = \frac{37\pi}{10}.$$

4) It follows immediately that $\bigtriangledown g = (y, x)$, so the weight function is

$$\sqrt{1 + \| \bigtriangledown g \|^2} = \sqrt{1 + (x^2 + y^2)}.$$

Then we compute the surface integral,

$$\int_{\mathcal{F}} x^2 \sqrt{1 + x^2 + y^2}\, dS = \int_E x^2 \sqrt{1 + x^2 + y^2} \cdot \sqrt{1 + x^2 + y^2}\, dx\, dy$$

$$= \int_E x^2 (1 + x^2 + y^2)\, dx\, dy = \int_0^{2\pi} \left\{ \int_0^1 \varrho^2 \cos^2 \varphi \cdot (1 + \varrho^2)\varrho\, d\varrho \right\} d\varphi$$

$$= \int_0^{2\pi} \cos^2 \varphi\, d\varphi \cdot \frac{1}{2} \int_0^1 t(1 + t)\, dt = \pi \cdot \frac{1}{2} \left[\frac{1}{2}\, t^2 + \frac{1}{3}\, t^3 \right]_0^1 = \frac{5\pi}{12}.$$
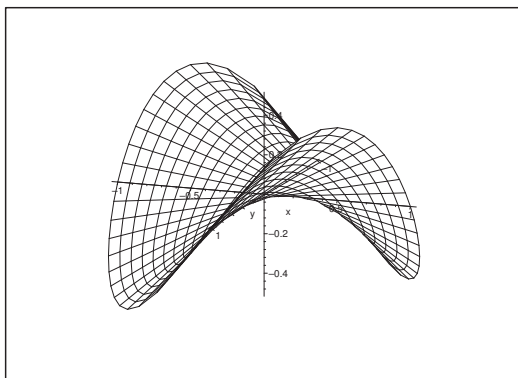
Figure 3: The surface of **Example 1.1.4**.

5) Here $\bigtriangledown g = \dfrac{2}{a}(x, -y)$, hence the weight is

$$\sqrt{1 + \parallel \bigtriangledown g \parallel^2} = \sqrt{1 + \frac{4}{a^2}(x^2 + y^2)} = \frac{1}{a}\sqrt{a^2 + 4(x^2 + y^2)}.$$
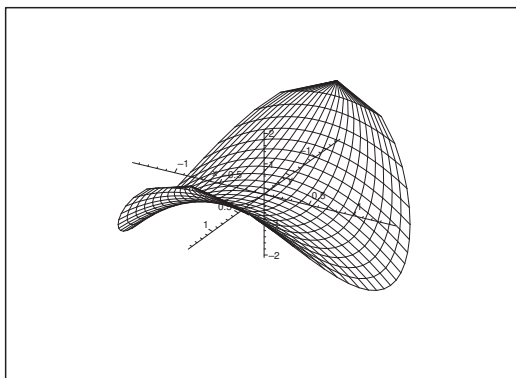
Figure 4: The surface of **Example 1.1.5, Example 1.1.6** and **Example 1.1.7**.

Then we get the surface integral,

$$
\begin{aligned}
\int_{\mathcal{F}} (a+z)\, dS &= \int_E \left(a + \frac{x^2 - y^2}{a} \right) \frac{1}{a}\sqrt{a^2 + 4(x^2 + y^2)}\, dx\, dy \\
&= \frac{1}{a^2} \int_E (a^2 + x^2 - y^2)\sqrt{a^2 + 4(x^2 + y^2)}\, dx\, dy \\
&= \frac{1}{a^2} \int_0^{2\pi} \left\{ \int_0^{\sqrt{2}a} (a^2 + \varrho^2[\cos^2\varphi - \sin^2\varphi])\sqrt{a^2 + 4\varrho^2}\, \varrho\, d\varrho \right\} d\varphi \\
&= 2\pi \int_0^{\sqrt{2}a} \sqrt{a^2 + 4\varrho^2} \cdot \varrho\, d\varrho + \frac{1}{a^2} \int_0^{2\pi} \cos 2\varphi\, d\varphi \int_0^{\sqrt{2}a} \varrho^2 \sqrt{a^2 + 4\varrho^2}\, \varrho\, d\varrho \\
&= \frac{2\pi}{8} \int_{\varrho=0}^{\sqrt{2}a} (a^2 + 4\varrho^2)^{\frac{1}{2}} d(a^2 + 4\varrho^2) + 0 = \frac{\pi}{4} \cdot \frac{2}{3} \left[(a^2 + 4\varrho^2)^{\frac{3}{2}} \right]_{\varrho=0}^{\sqrt{2}a} \\
&= \frac{\pi}{6} \left\{ (a^2 + 4 \cdot 2a^2)^{\frac{3}{2}} - a^3 \right\} = \frac{13\pi}{3}\, a^3.
\end{aligned}
$$

6) The surface is the same as in **Example 1.1.5**. Therefore, we get the weight function
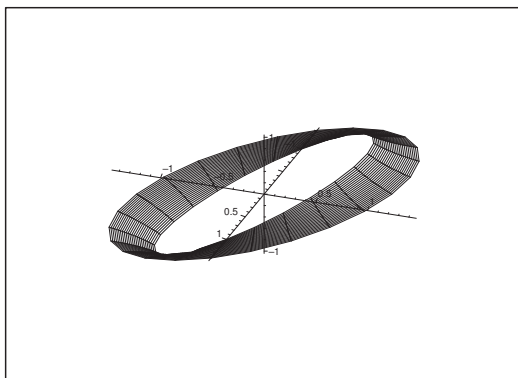
$$
\sqrt{1 + \| \bigtriangledown g \|^2} = \frac{1}{a}\sqrt{a^2 + 4(x^2 + y^2)},
$$

and the surface integral is

$$
\int_{\mathcal{F}} \frac{1}{\sqrt{a^2 + 4x^2 + 4y^2}}\, dS = \int_E \frac{1}{a}\, dx\, dy = \frac{1}{a}\, \text{area}(E) = \frac{1}{a} \cdot \pi \cdot 2a^2 = 2\pi a.
$$

7) The surface is the same as in **Example 1.1.5**, so the weight function is

$$
\sqrt{1 + \| \bigtriangledown g \|^2} = \frac{1}{a}\sqrt{a^2 + 4(x^2 + y^2)},
$$

Figure 5: The surface of **Example 1.1.10**.

and the surface integral becomes

$$
\int_{\mathcal{F}} \sqrt{a^2 + 4x^2 + 4y^2} \, dS = \int_E \frac{1}{a} \left(a^2 + 4(x^2 + y^2) \right) dx \, dy
$$

$$
= \frac{2\pi}{a} \int_0^{\sqrt{2}a} (a^2 + 4\varrho^2)\varrho \, d\varrho = \frac{2\pi}{a} \left[\frac{1}{2} a^2 \varrho^2 + \varrho^4 \right]_{\varrho=0}^{\sqrt{2}a}
$$

$$
= \frac{2\pi}{a} \left\{ \frac{1}{2} a^2 \cdot 2a^2 + 4a^4 \right\} = \frac{2\pi}{a} \cdot 5a^4 = 10\pi a^3.
$$

8) Here

$$
\bigtriangledown g = \left(-\frac{2xy}{(x^2+y^2)^2}, \frac{x^2-y^2}{(x^2+y^2)^2} \right) = \frac{1}{(x^2+y^2)^2} \left(-2xy, x^2 - y^2 \right),
$$

hence

$$
\| \bigtriangledown g \|^2 = \frac{1}{(x^2+y^2)^4} \left(4x^2y^2 + (x^2-y^2)^2 \right) = \frac{1}{(x^2+y^2)^2}.
$$

The surface integral is

$$
\begin{aligned}
\int_{\mathcal{F}} dS &= \int_E \frac{1}{x^2+y^2} \sqrt{1 + (x^2+y^2)^2} \, dx \, dy = 2\pi \int_0^{\sqrt{2}} \frac{\sqrt{1+\varrho^4}}{\varrho^2} \, \varrho \, d\varrho \\
&= \frac{2\pi}{4} \int_1^{\sqrt{2}} \frac{\sqrt{1+\varrho^4}}{\varrho^4} \cdot 4\varrho^3 \, d\varrho = \frac{\pi}{2} \int_1^4 \frac{1+t}{t} \, dt = \frac{\pi}{2} \int_{\sqrt{2}}^{\sqrt{5}} \frac{u \cdot 2u}{u^2-1} \, du \\
&= \pi \int_{\sqrt{2}}^{\sqrt{5}} \left\{ 1 + \frac{1}{2} \frac{1}{u-1} - \frac{1}{2} \frac{1}{u+1} \right\} du = \pi \left[u + \frac{1}{2} \ln \frac{u-1}{u+1} \right]_{\sqrt{2}}^{\sqrt{5}} \\
&= \pi \left\{ \sqrt{5} - \sqrt{2} + \frac{1}{2} \ln \left(\frac{\sqrt{5}-1}{\sqrt{5}+1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}-1} \right) \right\} \\
&= \pi \left\{ \sqrt{5} - \sqrt{2} + \ln \left(\frac{(\sqrt{5}-1)(\sqrt{2}+1)}{2} \right) \right\} \\
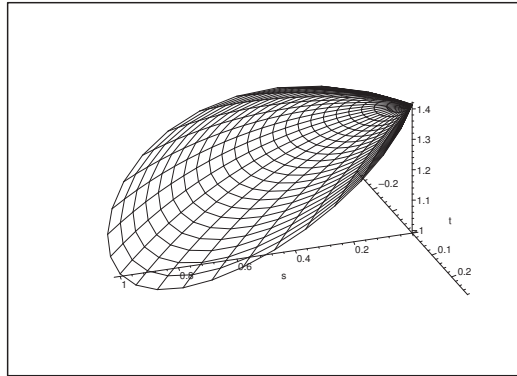&= \pi \{ \sqrt{5} - \sqrt{2} + \ln(\sqrt{5}-1) + \ln(\sqrt{2}+1) - \ln 2.
\end{aligned}
$$

Figure 6: The surface of **Example 1.1.11** for $a = 1$.

9) It follows from $g(x, y) = \sqrt{2a^2 - x^2 - y^2}$ that

$$\nabla g = \frac{1}{\sqrt{2a^2 - x^2 - y^2}} \, (-x - y),$$

hence

$$\sqrt{1 + \|\nabla g\|^2} = \sqrt{1 + \frac{x^2 + y^2}{2a^2 - x^2 - y^2}} = \frac{\sqrt{2} \cdot a}{\sqrt{2a^2 - x^2 - y^2}}.$$

If we use polar coordinates in the parameter domain, we get

$$
\begin{aligned}
\int_{\mathcal{F}} z^3 \, dS &= \int_E \left(\sqrt{2a^2 - x^2 - y^2}\right)^3 \cdot \frac{\sqrt{2} \cdot a}{\sqrt{2a^2 - x^2 - y^2}} \, dx \, dy \\
&= \sqrt{2}\, a \int_E (2a^2 - x^2 - y^2) \, dx \, dy = \sqrt{2}\, a \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\{ \int_0^{a\cos 2\varphi} (2a^2 - \varrho^2)\varrho \, d\varrho \right\} d\varphi \\
&= \sqrt{2}\, a \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[a^2 \varrho^2 - \frac{1}{4}\varrho^4 \right]_0^{a\cos 2\varphi} d\varphi = \sqrt{2}\, a^5 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\cos^2 2\varphi - \frac{1}{4}\cos^4 2\varphi \right) d\varphi \\
&= 2\sqrt{2}\, a^5 \int_0^{\frac{\pi}{4}} \left\{ \frac{1}{2} + \frac{1}{2}\cos 4\varphi - \frac{1}{4}\left(\frac{1 + \cos 4\varphi}{2} \right)^2 \right\} d\varphi \\
&= 2\sqrt{2}\, a^5 \left[\frac{1}{2}\varphi + \frac{1}{8}\sin 4\varphi \right]_0^{\frac{\pi}{4}} - \frac{2\sqrt{2}}{16} a^5 \int_0^{\frac{\pi}{4}} \left(1 + 2\cos 4\varphi + \cos^2 4\varphi\right) d\varphi \\
&= \frac{\sqrt{2}\pi}{4} a^5 - \frac{2\sqrt{2}}{16} \cdot \frac{\pi}{4} a^5 - \frac{2\sqrt{2}}{16} \cdot \frac{1}{2} a^5 \cdot \frac{\pi}{4} = \frac{\sqrt{2}\pi\, a^5}{64}(16 - 2 - 1) \\
&= \frac{13\sqrt{2}}{64} \pi\, a^5.
\end{aligned}
$$

Example 1.2 *Compute in each of the following cases the given surface integral over a cylinder surface $\mathcal{C}$, which is given by the plane curve $\mathcal{L}$ in the (X,Y)-plane, and the interval, in which z lies, when (x,y) is a point of the curve. Notice that $\mathcal{L}$ can either be given by an equation in rectangular or in polar coordinates, or by a parametric description.*

1) *The surface integral $\int_{\mathcal{C}}(y^2 z + x^2 z + y)\, dS$, where the curve $\mathcal{L}$ is given by $x^2 + y^2 = 2x$, and where $z \in \left[0, \sqrt{x^2 + y^2}\right]$.*

2) *The surface integral $\int_{\mathcal{C}} z^2\, dS$, where the curve $\mathcal{L}$ is given by $x^2 + y^2 = 4$, and where $z \in [-2, x]$.*

3) *The surface integral $\int_{\mathcal{C}}(z^2 + x^2)\, dS$, where the curve $\mathcal{L}$ is given by $x^2 + y^2 = 1$, and where $z \in [0, 2]$.*

4) *The surface integral $\int_{\mathcal{C}} z\, dS$, where the curve $\mathcal{L}$ is given by $y = x^2$ for $x \in [0,1]$, and where $z \in [0, x]$.*

5) *The surface integral $\int_{\mathcal{C}} z\, dS$, where the curve $\mathcal{L}$ is given by $\mathbf{r}(t) = (a\cos^3 t, a\sin^3 t)$ for $t \in \left[0, \frac{\pi}{2}\right]$, and where $z \in [0, y]$.*

6) *The surface integral $\int_{\mathcal{C}} \frac{1}{x}\, dS$, where the curve $\mathcal{L}$ is given by $\varrho = e^\varphi$ for $\varphi \in [0,1]$, and where $z \in [0, x]$.*

7) *The surface integral $\int_{\mathcal{C}} \frac{z}{x^2}\, dS$, where the curve $\mathcal{L}$ is given by $\varrho = a\cos^2 \frac{\varphi}{2}$ for $\varphi \in \left[0, \frac{\pi}{2}\right]$, and where $z \in [0, xy]$.*
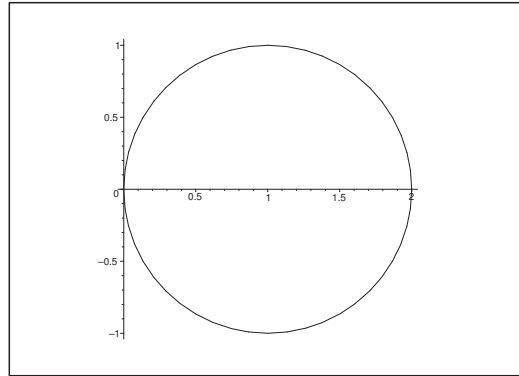
Figure 7: The curve $\mathcal{L}$ of **Example 1.2.1**.

8) The surface integral $\int_{\mathcal{C}} xz\,dS$, where the curve $\mathcal{L}$ is given by $x^2 + y^2 = ax$, and where $z \in \left[0, \sqrt{a^2 - x^2 - y^2}\right]$.

9) The surface integral $\int_{\mathcal{C}} dS$, where the curve $\mathcal{L}$ is given by $y = \ln \sin x$ for $x \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$, and where $z \in \left[0, \dfrac{\cos^2 x}{\sin x}\right]$.

10) The surface integral $\int_{\mathcal{C}} \cosh \dfrac{z}{a}\,dS$, where the curve $\mathcal{L}$ is given by $y = a \cosh \dfrac{x}{a}$ for $x \in [0, a]$, and where $z \in 0, x]$.

11) The surface integral $\int_{\mathcal{C}} z^2\,dS$, where the curve $\mathcal{L}$ is given by $y = x^3$ for $x \in [0, 1]$, and where $z \in [0, x]$.

A Surface integral over a cylinder surface.

D Reduce to a line integral by first integrating in the direction of the Z-axis. Find the line element and compute the line integral.

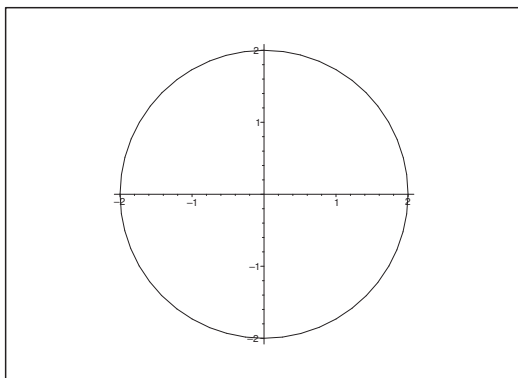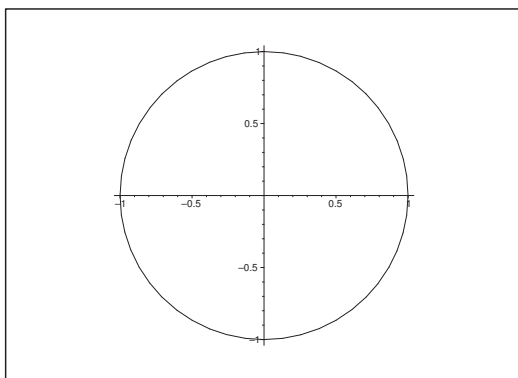**I** 1) The curve is the circle of centrum $(1, 0)$ and radius 1, thus in polar coordinates

$$\varrho(\varphi) = 2\cos\varphi, \qquad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

and the line element is

$$ds = \sqrt{\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2}\,d\varphi = \sqrt{4\cos^2\varphi + 4\sin^2\varphi}\,d\varphi = 2\,d\varphi.$$

Hence

$$\int_{\mathcal{C}} (y^2 z + x^2 z + y)\,dS = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{\int_0^{\varrho(\varphi)} \{z\varrho(\varphi)\sin\varphi\}\,dz\right\} 2\,d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{2}z^2\varrho(\varphi)^2 + \varrho(\varphi)z\sin\varphi\right]_{z=0}^{2\cos\varphi} \cdot 2\,d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{16\cos^4\varphi + 8\cos^2\varphi \cdot \sin\varphi\right\} d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4(1 + \cos 2\varphi)^2 d\varphi + 0 = 8\int_0^{\frac{\pi}{2}} \left(1 + 2\cos 2\varphi + \frac{1}{2} + \frac{1}{2}\cos 4\varphi\right) d\varphi = 8 \cdot \frac{3}{2} \cdot \frac{\pi}{2} = 6\pi.$$

Figure 8: The curve $\mathcal{L}$ of **Example 1.2.2**.

Figure 9: The curve $\mathcal{L}$ of **Example 1.2.3**.

2) The curve is the circle of centrum $(0,0)$ and radius 2. It is described in polar coordinates by

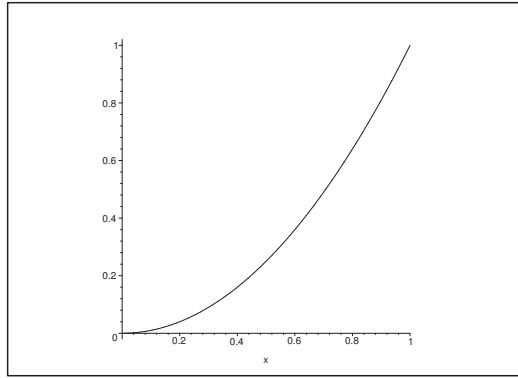$$\varrho = 2, \qquad \varphi \in [0, 2\pi],$$

thus the line element is

$$ds = \sqrt{\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2}\, d\varphi = 2\, d\varphi.$$

Hence

$$\begin{aligned}
\int_{\mathcal{C}} z^2\, dS &= \int_0^{2\pi} \left\{ \int_{-2}^{2\cos\varphi} z^2\, dz \right\} 2\, d\varphi = \frac{2}{3} \int_0^{2\pi} \left\{ 8\cos^3\varphi - (-2)^3 \right\} d\varphi \\
&= \frac{16}{3} \int_0^{2\pi} \left\{ \cos^3\varphi + 1 \right\} d\varphi = \frac{32\pi}{3} + 0 = \frac{32\pi}{3}.
\end{aligned}$$

3) The curve is the unit circle given in polar coordinates by

$$\varrho = 1, \qquad \varphi \in [0, 2\pi].$$

Figure 10: The curve $\mathcal{L}$ of **Example 1.2.4**.

Thus $ds = d\varphi$, and

$$
\begin{aligned}
\int_{\mathcal{C}} (z^2 + x^2)\, dS &= \int_0^{2\pi} \left\{ \int_0^2 (z^2 + \cos^2 \varphi)\, dz \right\} d\varphi \\
&= \frac{8}{3} \cdot 2\pi + 2 \int_0^{2\pi} \cos^2 \varphi\, d\varphi = \frac{8}{3} \cdot 2\pi + 2\pi = \frac{22\pi}{3}.
\end{aligned}
$$

4) The curve is an arc of a parabola. It follows by putting $y = g(x) = x^2$ that the line element is

$$
ds = \sqrt{1 + g'(x)^2}\, dx = \sqrt{1 + 4x^2}\, dx,
$$

thus

$$
\int_{\mathcal{C}} z\, dS = \int_0^1 \left\{ \int_0^x z\, dz \right\} \sqrt{1 + 4x^2}\, dx = \frac{1}{2} \int_0^1 x^2 \sqrt{1 + 4x^2}\, dx.
$$

Then we get by the substitution $x = \frac{1}{2} \sinh t$, $t = \mathrm{Arsinh}(2t)$ that

$$
\begin{aligned}
\int_{\mathcal{C}} z\, dS &= \frac{1}{2} \int_0^{\mathrm{Arsinh}\, 2} \frac{1}{4} \sinh^2 t \cdot \cosh t \cdot \frac{1}{2} \cosh t\, dt = \frac{1}{16} \int_0^{\mathrm{Arsinh}\, 2} \left(\frac{1}{2} \sinh 2t \right)^2 dt \\
&= \frac{1}{64} \int_0^{\mathrm{Arsinh}\, 2} \frac{1}{2}(\cosh 4t - 1)\, dt = \frac{1}{512} \left[\sinh 4t \right]_0^{\mathrm{Arsinh}\, 2} - \frac{1}{128} \mathrm{Arsinh}\, 2 \\
&= \frac{1}{512} \left[4 \sinh t \cdot \sqrt{1 + \sinh^2 t} \cdot (1 + 2\sinh^2 t) \right]_0^{\mathrm{Arsinh}\, 2} - \frac{1}{128} \ln(2 + \sqrt{5}) \\
&= \frac{1}{128} \cdot 2\sqrt{5} \cdot (1 + 2 \cdot 4) - \frac{1}{128} \ln(2 + \sqrt{5}) = \frac{9\sqrt{5}}{64} - \frac{1}{128} \ln(2 + \sqrt{5}).
\end{aligned}
$$

5) We have in the given interval, $\cos t \cdot \sin t \geq 0$, so we do not need the absolute sign in the latter equality,

$$
\begin{aligned}
\|\mathbf{r}'(t)\| &= a\sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} \\
&= 3a\sqrt{cos^2 t \left\{ \cos^2 t \sin^2 t \right\} + \sin^2 t \left\{ \cos^2 t \sin^2 t \right\}} = 3a\, \cos t \sin t,
\end{aligned}
$$
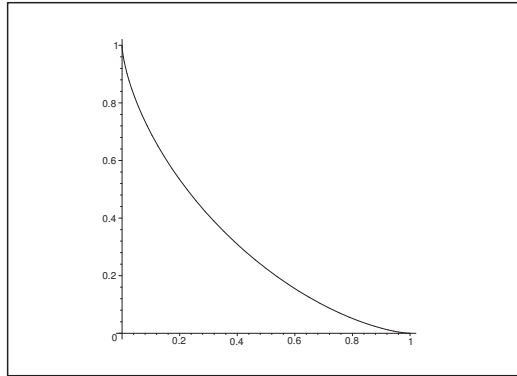
Figure 11: The curve $\mathcal{L}$ of **Example 1.2.5** for $a = 1$.
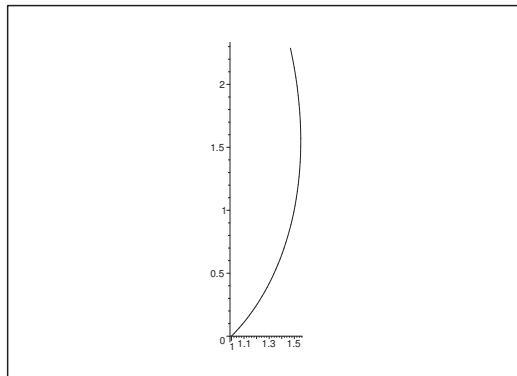
Figure 12: The curve $\mathcal{L}$ of **Example 1.2.6**.

thus the line element becomes

$$ds = 3a\,\cos t\,\sin t\,dt, \qquad t \in \left[0, \frac{\pi}{2}\right].$$

Then

$$
\begin{aligned}
\int_{\mathcal{C}} z\,dS &= \int_0^{\frac{\pi}{2}} \left\{ \int_0^{a\sin^3 t} z\,dz \right\} 3a\,\cos t\,\sin t\,dt = \frac{3a^2}{2} \int_0^{\frac{\pi}{2}} \sin^7 t\,\cos t\,dt \\
&= \frac{3}{16}\,a^3 \left[\sin^8 t\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{16}.
\end{aligned}
$$

6) The line element along the curve is

$$ds = \sqrt{\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2}\,d\varphi = \sqrt{2}\,e^{\varphi}\,d\varphi, \qquad \varphi \in [0,1],$$

and we get the surface integral

$$\int_{\mathcal{C}} \frac{1}{x}\,dS = \int_0^1 \frac{1}{x} \cdot x\,\sqrt{2}\,e^{\varphi}\,d\varphi = \sqrt{2}(e-1).$$

7) The line element is

$$
\begin{aligned}
ds &= \sqrt{\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2}\, d\varphi = \sqrt{a^2 \cos^4 \frac{\varphi}{2} + a^2 \left(-2 \cos \frac{\varphi}{2} \cdot \sin \frac{\varphi}{2} \cdot \frac{1}{2}\right)^2}\, d\varphi \\
&= a \cos \frac{\varphi}{2}\, d\varphi \qquad \text{for } \varphi \in \left[0, \frac{\pi}{2}\right],
\end{aligned}
$$

Figure 13: The curve $\mathcal{L}$ of **Example 1.2.7** for $a = 1$.

hence

$$
\begin{aligned}
\int_{\mathcal{C}} \frac{z}{x^2}\, dS &= \int_0^{\frac{\pi}{2}} \frac{1}{x^2} \left\{ \int_0^{xy} z\, dz \right\} a\, \cos\frac{\varphi}{2}\, d\varphi = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{(xy)^2}{x^2} \cdot a\, \cos\frac{\varphi}{2}\, d\varphi \\
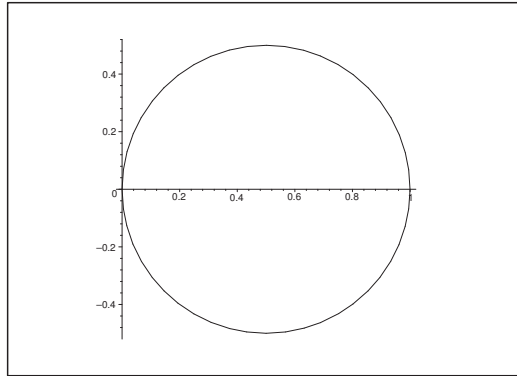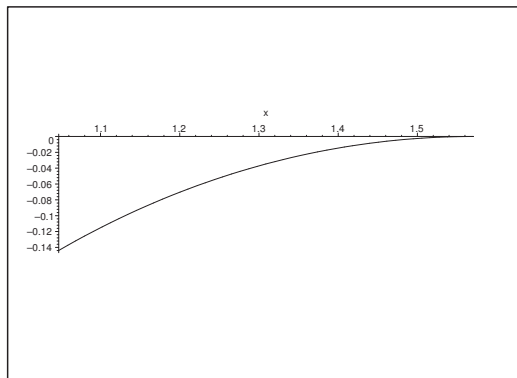&= \frac{a}{2} \int_0^{\frac{\pi}{2}} \varrho(\varphi)^2 \sin^2\varphi \cdot \cos\frac{\varphi}{2}\, d\varphi = \frac{a}{2} \int_0^{\frac{\pi}{2}} a^2 \cos^4\frac{\varphi}{2} \cdot 4\sin^2\frac{\varphi}{2} \cdot \cos^2\frac{\varphi}{2} \cdot \cos\frac{\varphi}{2}\, d\varphi \\
&= 2a^3 \int_0^{\frac{\pi}{2}} \cos^6\frac{\varphi}{2} \cdot \sin^2\frac{\varphi}{2} \cdot \cos\frac{\varphi}{2}\, d\varphi \\
&= 4a^3 \int_0^{\frac{\pi}{2}} \left\{ 1 - \sin^2\frac{\varphi}{2} \right\}^3 \cdot \sin^2\frac{\varphi}{2} \cdot \left(\frac{1}{2} \cos\frac{\varphi}{2} \right)\, d\varphi \\
&= 4a^3 \int_{\varphi=0}^{\frac{\pi}{2}} \left\{ \sin^2\frac{\varphi}{2} - 3\sin^4\frac{\varphi}{2} + 3\sin^6\frac{\varphi}{2} - \sin^8\frac{\varphi}{2} \right\} d\left(\sin\frac{\varphi}{2} \right) \\
&= 4a^3 \left[\frac{1}{3} t^3 - \frac{3}{5} t^5 + \frac{3}{7} t^7 - \frac{1}{9} t^9 \right]_0^{\frac{1}{\sqrt{2}}} = \frac{4a^3}{2\sqrt{2}} \left\{ \frac{1}{3} - \frac{3}{5}\cdot\frac{1}{2} + \frac{3}{7}\cdot\frac{1}{4} - \frac{1}{9}\cdot\frac{1}{8} \right\} \\
&= \frac{a^3\sqrt{2}}{2520}\left(840 - 756 + 270 - 35\right) = \frac{319\sqrt{2}}{2520}\, a^3.
\end{aligned}
$$

8) The curve is in polar coordinates given by

$$\varrho = a\, \cos\varphi, \qquad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right],$$

thus

$$ds = \sqrt{ \varrho^2 + \left(\frac{d\varrho}{d\varphi} \right)^2 }\, d\varphi = a\, d\varphi,$$

Figure 14: The curve $\mathcal{L}$ of **Example 1.2.8** for $a = 1$.

Figure 15: The curve $\mathcal{L}$ of **Example 1.2.9**.

and

$$
\begin{aligned}
\int_{\mathcal{C}} xz \, dS &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^{\sqrt{a^2 - a^2 \cos^2 \varphi}} a \, \cos^2 \varphi \cdot z \, dz \right\} a \, d\varphi \\
&= \frac{a^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \varphi \, (1 - \cos^2 \varphi) a^2 d\varphi = \frac{a^4}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} \sin 2\varphi \right)^2 d\varphi \\
&= \frac{a^4}{2} \cdot \frac{1}{4} \cdot 2 \int_0^{\frac{\pi}{2}} \sin^2 2\varphi \, d\varphi = \frac{a^4}{8} \int_0^{\frac{\pi}{2}} (1 - \cos 4\varphi) \, d\varphi = \frac{a^4 \pi}{16}.
\end{aligned}
$$

9) We derive from $y = g(x) = \ln \sin x$, $x \in \left[\dfrac{\pi}{3}, \dfrac{\pi}{2} \right]$ that the line element is

$$
ds = \sqrt{1 + g'(x)^2} \, dx = \sqrt{1 + \left(\frac{\cos x}{\sin x} \right)^2} \, dx = \frac{1}{\sin x} \, dx,
$$

and thus

$$
\begin{aligned}
\int_{\mathcal{C}} dS &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x} \cdot \frac{1}{\sin x}\, dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1}{\sin^2 x} - 1 \right) dx \\
&= \left[-\cot x - x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \cot \frac{\pi}{3} - \frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{\pi}{6}.
\end{aligned}
$$

Figure 16: The curve $\mathcal{L}$ of **Example 1.2.10** for $a = 1$.
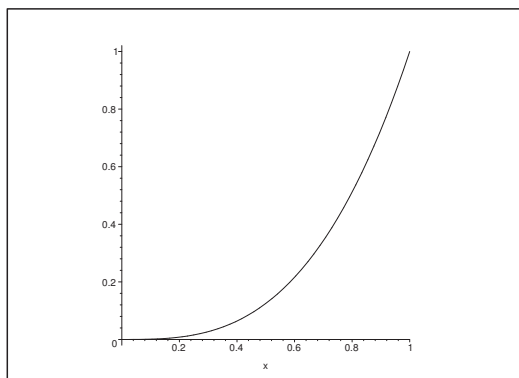
Figure 17: The curve $\mathcal{L}$ of **Example 1.2.11**.

10) When the curve is given by $y = g(x) = a \cosh \dfrac{x}{a}$, we obtain the line element

$$ds = \sqrt{1 + g'(x)^2}\, dx = \sqrt{1 + \sinh^2 \frac{x}{a}}\, dx = \cosh \frac{x}{a}\, dx,$$

so

$$
\begin{aligned}
\int_{\mathcal{C}} \cosh \frac{z}{a}\, dS &= \int_0^a \left\{ \int_0^x \cosh \frac{z}{a}\, dz \right\} \cdot \cosh \frac{x}{a}\, dx \\
&= a \int_0^a \sinh \frac{x}{a} \cdot \cosh \frac{x}{a}\, dx = \frac{a^2}{2} \cdot \sinh^2 1 \\
&= \frac{a^2}{2} \left(\frac{e - e^{-1}}{2} \right)^2 = \frac{a^2}{8e^2} \left(e^2 - 1 \right)^2 = \frac{a^2}{8e^2} \left(e^4 - 2e^2 + 1 \right).
\end{aligned}
$$

11) For the curve given by $y = g(x) = x^3$, the line element is

$$ds = \sqrt{1 + g'(x)^2}\, dx = \sqrt{1 + 9x^4}\, dx,$$

hence

$$
\begin{aligned}
\int_{\mathcal{C}} z^2 \, dS &= \int_0^1 \left\{ \int_0^x z^2 \, dz \right\} \sqrt{1 + 9x^4} \, dx = \frac{1}{3} \int_0^1 \sqrt{1 + 9x^4} \cdot x^3 \, dx \\
&= \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{9} \int_0^1 \sqrt{1 + 9x^4} \cdot 9 \, d\left(x^4\right) = \frac{1}{108} \cdot \frac{2}{3} \left[\left(\sqrt{1 + 9x^4}\right)^3 \right]_0^1 \\
&= \frac{1}{162} \left(10\sqrt{10} - 1\right).
\end{aligned}
$$

Figure 18: The meridian curve $\mathcal{M}$ of **Example 1.3.1** for $a = 1$.

Example 1.3 *Compute in each of the following cases the given surface integral over a surface of revolution $\mathcal{O}$ which is given by a meridian curve $\mathcal{M}$ in the meridian half plane, in which ϱ and z are rectangular coordinates.*

1) *The surface integral $\int_{\mathcal{O}}(x^2 + y^2)\, dS$, where the meridian curve $\mathcal{M}$ is given by $z = \dfrac{\varrho^2}{2a}$ for $\varrho \leq a$.*

2) *The surface integral $\int_{\mathcal{O}}(x^2 + y^2)\, dS$, where the meridian curve $\mathcal{M}$ is given by $z = \dfrac{h\varrho}{a}$ for $\varrho \leq a$.*

3) *The surface integral $\int_{\mathcal{O}} dfrace^z \varrho\, dS$, where the meridian curve $\mathcal{M}$ is given by $z = \ln \in \varrho$ for $\varrho \in \left[\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right]$.*

4) *The surface integral $\int_{\mathcal{O}} x^2\, dS$, where the meridian curve $\mathcal{M}$ is given by $z^2 + \varrho^2 = az$.*

5) *The surface integral $\int_{\mathcal{O}} |x| e^{-x}\, dS$, where the meridian curve $\mathcal{M}$ is given by $z = -\ln\cos\varrho$ for $\varrho \in \left[0, \dfrac{\pi}{3}\right]$.*

6) *The surface integral $\int_{\mathcal{O}} \dfrac{y^2}{z}\, dS$, where the meridian curve $\mathcal{M}$ is given by $z = a\cosh\dfrac{\varrho}{a}$ for $\varrho \in [0, a]$.*

A Surface integral over a surface of revolution.

D Use either semi polar or spherical coordinates and the area element $\varrho\, d\varphi\, ds$, where ds is the curve element, i.e. if e.g. $z = g(\varrho)$, then

$$ds = \sqrt{1 + g'(\varrho)^2}\, d\varrho,$$

and similarly.
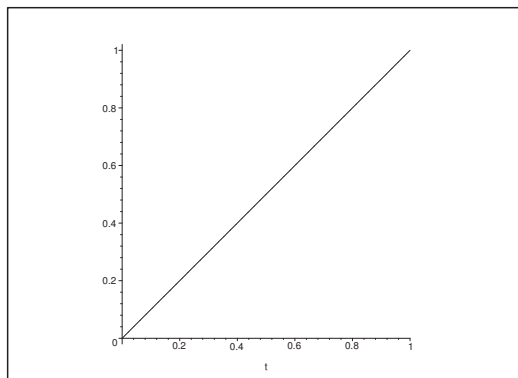
Figure 19: The meridian curve $\mathcal{M}$ of **Example 1.3.2** for $a = 1$ and $h = 1$.

I 1) Here $ds = \sqrt{1 + \left(\dfrac{\varrho}{a}\right)^2}\, d\varrho$, hence

$$
\begin{aligned}
\int_{\mathcal{O}} (x^2 + y^2)\, dS &= \int_0^{2\pi} \left\{ \int_0^a \varrho^2 \cdot \varrho \sqrt{1 + \left(\frac{\varrho}{a}\right)^2}\, d\varrho \right\} d\varphi = 2\pi \cdot \frac{a^4}{2} \int_0^1 t\sqrt{1+t}\, dt \\
&= \pi a^4 \int_0^1 \left\{ (1+t)^{\frac{3}{2}} - (1+t)^{\frac{1}{2}} \right\} dt = \pi a^4 \left[\frac{2}{5}(1+t)^{\frac{5}{2}} - \frac{2}{3}(1+t)^{\frac{3}{2}} \right]_0^1 \\
&= \pi a^4 \left\{ \frac{2}{5}\left(2^{\frac{5}{2}} - 1\right) - \frac{2}{3}\left(2^{\frac{3}{2}} - 1\right) \right\} = \frac{\pi a^4}{15}\{6(4\sqrt{2} - 1) - 10(2\sqrt{2} - 1)\} \\
&= \frac{\pi a^4}{15}\{24\sqrt{2} - 6 - 20\sqrt{2} + 10\} = \frac{\pi a^4}{15}\{4\sqrt{2} + 4\} = \frac{4\pi a^4}{15}(\sqrt{2} + 1).
\end{aligned}
$$

2) Here

$$
ds = \sqrt{1 + \frac{h^2}{a^2}}\, d\varrho = \frac{1}{a}\sqrt{a^2 + h^2}\, d\varrho,
$$

hence

$$
\begin{aligned}
\int_{\mathcal{O}} (x^2 + y^2)\, dS &= \int_0^{2\pi} \left\{ \int_0^a \varrho^2 \cdot \varrho \cdot \frac{1}{a}\sqrt{a^2 + h^2}\, d\varrho \right\} d\varphi \\
&= 2\pi \cdot \frac{1}{a}\sqrt{a^2 + h^2} \cdot \frac{1}{4}a^4 = \frac{\pi}{2}a^3 \sqrt{a^2 + h^2}.
\end{aligned}
$$

3) From $z = \ln \sin \varrho$ follows that $\dfrac{dz}{d\varrho} = \dfrac{\cos \varrho}{\sin \varrho}$, hence

$$
\sqrt{1 + \left(\frac{dz}{d\varrho}\right)^2} = \sqrt{1 + \frac{\cos^2 \varrho}{\sin^2 \varrho}} = \frac{1}{|\sin \varrho|} = \frac{1}{\sin \varrho} \quad \text{for } \varrho \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right].
$$

The area element is

$$
\varrho\, d\varphi\, ds = \frac{\varrho}{\sin \varrho}\, d\varrho\, d\varphi = dS,
$$
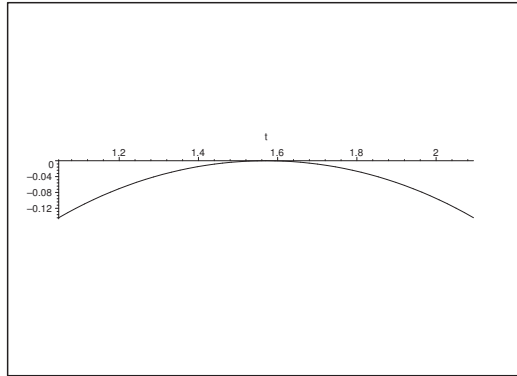
Figure 20: The meridian curve $\mathcal{M}$ of **Example 1.3.3**.
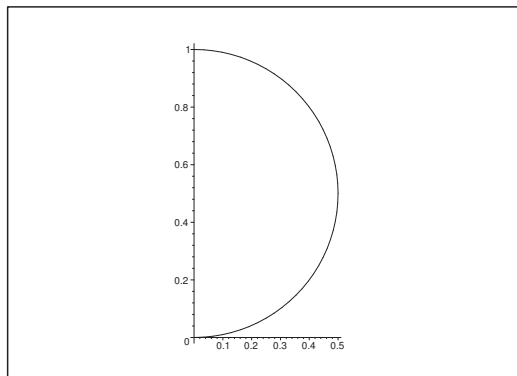
Figure 21: The meridian curve $\mathcal{M}$ of **Example 1.3.4** and **Example 1.3.5** for $a = 1$.

hence by insertion

$$\int_{\mathcal{O}} \frac{e^z}{\varrho}\, dS = \int_0^{2\pi} \left\{ \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin \varrho}{\varrho} \cdot \frac{\varrho}{\sin \varrho}\, d\varrho \right\} d\varphi = 2\pi \left(\frac{2\pi}{3} - \frac{\pi}{3} \right) = \frac{2\pi^2}{3}.$$

4) The figure shows that the meridian curve is a half circle of radius $\dfrac{a}{2}$. Hence, the integral $\int_{\mathcal{O}} dS$ is equal to the surface area of the sphere, i.e.

$$\int_{\mathcal{O}} dS = 4\pi \left(\frac{a}{2} \right)^2 = \pi a^2$$

where we have used the result of **Example 1.3.6** with $a = b$.

ALTERNATIVELY,

$$\varrho = \sqrt{ \left(\frac{a}{2} \right)^2 - \left(z - \frac{a}{2} \right)^2 }, \qquad \text{for } z \in [0, a],$$

in rectangular coordinates, so

$$ds = \sqrt{1 + \frac{\left(z - \frac{a}{2}\right)^2}{\left(\frac{a}{2}\right)^2 - \left(z - \frac{a}{2}\right)^2}} \, dz = \frac{a}{2} \frac{1}{\sqrt{\left(\frac{a}{2}\right)^2 - \left(z - \frac{a}{2}\right)^2}} \, dz.$$

Thus

$$\int_{\mathcal{O}} dS = 2\pi \int_0^a \sqrt{\left(\frac{a}{2}\right)^2} \cdot \frac{a}{2} \cdot \frac{1}{\sqrt{\left(\frac{a}{2}\right)^2 - \left(z - \frac{a}{2}\right)^2}} \, dz = 2\pi \cdot \frac{a}{2} \cdot a = \pi a^2.$$

ALTERNATIVELY we have $r = a\cos\theta$, $\theta \in \left[0, \frac{\pi}{2}\right]$, in spherical coordinates, and $\varrho = r\sin\theta = a\sin\theta\cos\theta$, and

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = a \, d\theta,$$

and we get

$$\int_{\mathcal{O}} dS = 2\pi \int_0^{\frac{\pi}{2}} a \sin\theta \, \cos\theta \cdot a \, d\theta = a^2\pi \left[\sin^2\theta\right]_0^{\frac{\pi}{2}} = a^2\pi.$$

Figure 22: The meridian curve $\mathcal{M}$ of **Example 1.3.6**.

5) From $x = \varrho \cos\varphi$ in semi polar coordinates we get from **Example 1.3.4** that

$$
\int_{\mathcal{O}} x^3 \, dS = \int_0^{2\pi} \left\{ \int_0^a \left\{ (\tfrac{a}{2})^2 - (z - \tfrac{a}{2})^2 \right\} \cos^2\varphi \cdot \frac{a}{2} \cdot \frac{\sqrt{(\tfrac{a}{2})^2 - (z - \tfrac{a}{2})^2}}{\sqrt{(\tfrac{a}{2})^2 - (z - \tfrac{a}{2})^2}} \, dz \right\} d\varphi
$$

$$
= \frac{a}{2} \int_0^{2\pi} \cos^2\varphi \, d\varphi \int_0^a (az - z^2) \, dz = \frac{a}{2} \cdot \pi \left[\frac{a}{2} z^2 - \frac{1}{3} z^3 \right]_0^a = \frac{a^4 \pi}{12}.
$$

ALTERNATIVELY,

$$
x = r \, \sin\theta \, \cos\varphi = a \cos\theta \, \cos\varphi
$$

in spherical coordinates, cf. **Example 1.3.4**, so accordingly

$$
\int_{\mathcal{O}} dS = \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}} a^2 \cos^2\theta \, \sin^2\theta \, \cos^2\varphi \cdot a \, \sin\theta \, \cos\theta \, a \, d\theta \right\} d\varphi
$$

$$
= a^4 \int_0^{2\pi} \cos^2\varphi \, d\varphi \int_0^{\frac{\pi}{2}} \sin^3\theta \cdot (1 - \sin^2\theta) \cos\theta \, d\theta = a^4 \pi \left[\frac{1}{4} - \frac{1}{6} \right] = \frac{a^4 \pi}{12}.
$$

6) As

$$
ds = \sqrt{1 + \left(\frac{\sin\varrho}{\cos\varrho} \right)^2} \, d\varrho = \frac{1}{\cos\varrho} \, d\varrho, \quad \text{for } \varrho \in \left[0, \frac{\pi}{3} \right],
$$

we get

$$
\int_{\mathcal{O}} |x| e^{-z} \, dS = \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{3}} \varrho |\cos\varphi| \cdot \cos\varrho \cdot \frac{\varrho}{\cos\varrho} \, d\varrho \right\} d\varphi
$$

$$
= 4 \int_0^{\frac{\pi}{2}} \cos\varphi \, d\varphi \cdot \int_0^{\frac{\pi}{3}} \varrho^2 \, d\varrho = 4 \cdot \frac{1}{3} \left(\frac{\pi}{3} \right)^3 = \frac{4\pi^3}{81}.
$$

7) We get from $z = g(\varrho) = a \cosh \dfrac{\varrho}{a}$ that $g'(\varrho) = \sinh \dfrac{\varrho}{a}$ and

$$
ds = \sqrt{1 + g'(\varrho)^2} \, d\varrho = \sqrt{1 + \sinh^2 \frac{\varrho}{a}} \, d\varrho = \cosh \frac{\varrho}{a} \, d\varrho,
$$
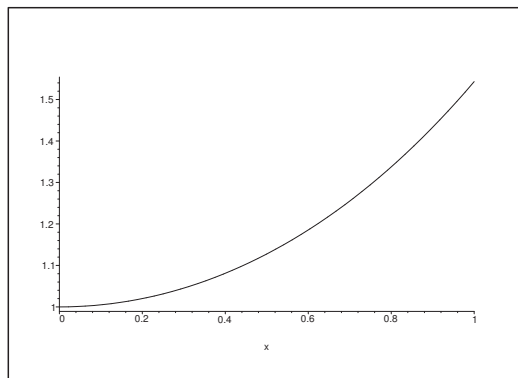
Figure 23: The meridian curve $\mathcal{M}$ of **Example 1.3.7** for $a = 1$.

hence

$$
\int_{\mathcal{O}} \frac{y^2}{z} \, dS = \int_0^{2\pi} \left\{ \int_0^a \frac{\varrho^2 \sin^2 \varphi}{a \cosh \dfrac{\varrho}{a}} \cdot \varrho \cdot \cosh \frac{\varrho}{a} \, d\varrho \right\} d\varphi
$$

$$
= \frac{1}{a} \int_0^{2\pi} \sin^2 \varphi \, d\varphi \cdot \int_0^a \varrho^3 \, d\varrho = \frac{1}{a} \cdot \pi \cdot \frac{1}{4} \, a^4 = \frac{\pi a^3}{4}.
$$

Example 1.4 *Calculate in each of the following cases the given surface integral over the surface given by a parametric description*

$$
\mathcal{F} = \left\{ x \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{r}(u, v), \; (u, v) \in E \right\}.
$$

First find the normal vector of the surface $\mathbf{N}(u, v)$.

1) *The surface integral* $\int_{\mathcal{F}} xz^2 \, dS$, *where the surface* $\mathcal{F}$ *is given by*

$$
\mathbf{x} = \mathbf{r}(u, v) = (u \cos v, u \sin v, hv), \quad \text{for } 0 \leq u \leq 1, \, 0 \leq v \leq 2\pi.
$$

2) *The surface integral* $\int_{\mathcal{F}} z^2 \, dS$, *where the surface* $\mathcal{F}$ *is given by*

$$
\mathbf{x} = \mathbf{r}(u, v) = (\sqrt{u} \cos v, \sqrt{u} \sin v, e^v) \quad \text{for } 1 \leq u \leq 2, \, \frac{\ln u}{2} \leq v \leq \frac{\ln(2u)}{2}.
$$

3) *The surface integral* $\int_{\mathcal{F}} (x^2 + y^2) \, dS$, *where the surface* $\mathcal{F}$ *is given by*

$$
\mathbf{x} = \mathbf{r}(u, v) = \left(\sqrt{u} \cos v, \sqrt{u} \sin v, v^{\frac{3}{2}} \right) \quad \text{for } 1 \leq u \leq 2, \, 0 \leq v \leq u.
$$

4) *The surface integral* $\int_{\mathcal{F}} (x^3 + 2z - 3xy) \, dS$, *where the surface* $\mathcal{F}$ *is given by*

$$
\mathbf{x} = \mathbf{r}(u, v) = (u + v, u^2 + v^2, u^3 + v^3) \quad \text{for } u + v \leq 0, \, u^2 + v^2 \leq 5.
$$

A Surface integrals, where the surface is given by a parametric description.

D First find the normal vector $\mathbf{N}(u,v)$. Then compute the weight function $\|\mathbf{N}(u,v)\|$ as a function of the parameters $(u,v) \in E$.

I 1) The normal vector is

$$\mathbf{N}(u,v) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & h \end{vmatrix} = (h\sin v, -h\cos v, u),$$

and we find accordingly the weight function

$$\|\mathbf{N}(u,v)\| = \sqrt{h^2 + u^2}.$$

Then we get the following reduction of the surface integral,

$$
\begin{aligned}
\int_{\mathcal{F}} xz^2\, dS &= \int_0^1 \left\{ \int_0^{2\pi} u\cos v \cdot h^2 v^2 \sqrt{h^2+u^2}\, dv \right\} du = h^2 \int_0^1 u\sqrt{h^2+u^2}\, du \cdot \int_0^1 v^2 \cos v\, dv \\
&= h^2 \left[\frac{1}{2} \cdot \frac{2}{3}(h^2+u^2)^{\frac{3}{2}} \right]_0^1 \cdot \left[v^2\sin v + 2v\cos v - 2\sin v \right]_0^{2\pi} \\
&= \frac{1}{3} h^2 \left\{ (h^2+1)^{\frac{3}{2}} - h^3 \right\} \cdot 4\pi = \frac{4\pi}{3} h^2 \left\{ (h^2+1)\sqrt{h^2+1} - h^3 \right\}.
\end{aligned}
$$

2) The normal vector is

$$\mathbf{N}(u,v) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{1}{2}\dfrac{1}{\sqrt{u}}\cos v & \dfrac{1}{2}\dfrac{1}{\sqrt{u}}\sin v & 0 \\ -\sqrt{u}\sin v & \sqrt{u}\cos v & e^v \end{vmatrix} = \left(\frac{1}{2}\frac{e^v}{\sqrt{u}}\sin v, -\frac{1}{2}\frac{e^v}{\sqrt{u}}\cos v, \frac{1}{2} \right),$$

so the weight function becomes

$$\|\mathbf{N}(u,v)\| = \sqrt{\frac{1}{4}\frac{e^{2v}}{u} + \frac{1}{4}} = \frac{1}{2\sqrt{u}}\sqrt{e^{2v}+u}.$$

Then we have the following reduction of the surface integral

$$
\begin{aligned}
\int_{\mathcal{F}} z^2\, dS &= \int_1^2 \left\{ \int_{\frac{1}{2}\ln u}^{\frac{1}{2}\ln(2u)} e^{2v} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{u}} \cdot \sqrt{e^{2v}+u}\, dv \right\} du \\
&= \frac{1}{2} \cdot \frac{1}{2} \int_1^2 \left\{ \int_{v=\frac{1}{2}\ln u}^{\frac{1}{2}\ln(2u)} \frac{1}{\sqrt{u}}\sqrt{e^{2v}+u}\, d\left(e^{2v}\right) \right\} du \\
&= \frac{1}{4} \int_1^2 \frac{1}{\sqrt{u}} \cdot \frac{2}{3} \left[(t+u)^{\frac{3}{2}} \right]_{t=u}^{t=2u} du = \frac{1}{6} \int_1^2 \frac{1}{\sqrt{u}} \left\{ (3u)^{\frac{3}{2}} - (2u)^{\frac{3}{2}} \right\} du \\
&= \frac{1}{6}(3\sqrt{3} - 2\sqrt{2}) \int_1^2 u\, du = \frac{1}{12}(3\sqrt{3} - 2\sqrt{2}\left[u^2\right]_1^2 = \frac{1}{4}(3\sqrt{3} - 2\sqrt{2}).
\end{aligned}
$$

3) The normal vector is

$$
\mathbf{N}(u,v) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} =
\begin{vmatrix}
\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\[2mm]
\dfrac{1}{2}\,\dfrac{1}{\sqrt{u}}\,\cos v & \dfrac{1}{2}\,\dfrac{1}{\sqrt{u}}\,\sin v & 0 \\[4mm]
-\sqrt{u}\,\sin v & \sqrt{u}\,\cos v & \dfrac{3}{2}\,\sqrt{v}
\end{vmatrix}
= \left(\frac{3}{4}\,\sqrt{\frac{v}{u}}\,\sin v, \; -\frac{3}{4}\,\sqrt{\frac{v}{u}}\,\cos v, \; \frac{1}{2} \right),
$$

and the weight function is

$$
\|\mathbf{N}(u,v)\| = \sqrt{\frac{9}{16}\,\frac{v}{u} + \frac{1}{4}} = \frac{3}{4}\,\sqrt{\frac{v}{u} + \frac{4}{9}}.
$$

Finally, we get the following reduction of the surface integral

$$
\begin{aligned}
\int_{\mathcal{F}} (x^2 + y^2)\, dS &= \int_1^2 \left\{ \int_0^u \left(u \cos^2 + u \sin^2 v \right) \cdot \frac{3}{4} \sqrt{\frac{v}{u} + \frac{4}{9}}\, dv \right\} du \\
&= \frac{3}{4} \int_1^2 \left\{ \int_0^u u \sqrt{\frac{v}{u} + \frac{4}{9}}\, dv \right\} du = \frac{3}{4} \int_1^2 \left\{ \int_0^u \sqrt{v + \frac{4}{9} u} \cdot \sqrt{u}\, dv \right\} du \\
&= \frac{3}{4} \cdot \frac{2}{3} \int_1^2 \sqrt{u} \left[\left(v + \frac{4}{9} u \right)^{\frac{3}{2}} \right]_0^u du = \frac{1}{2} \int_1^2 \sqrt{u} \left\{ \left(\frac{13}{9} u \right)^{\frac{3}{2}} - \left(\frac{4}{9} u \right)^{\frac{3}{2}} \right\} du \\
&= \frac{1}{2} \cdot \frac{1}{27} \left(13\sqrt{13} - 8 \right) \int_1^2 u^2\, du = \frac{7}{162} \left(13\sqrt{13} - 8 \right).
\end{aligned}
$$

4) The normal vector is

$$
\begin{aligned}
\mathbf{N}(u, v) &= \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 2u & 3u^2 \\ 1 & 2v & 3v^2 \end{vmatrix} = \left(6uv^2 - 6u^2 v,\, 3u^2 - 3v^2,\, 2v - 2u \right) \\
&= \left(6uv(vu),\, 3(u+v)(u-v),\, 2(v-u) \right) = (v-u)(6uv,\, -3(u+v),\, 2).
\end{aligned}
$$

Hence the weight function

$$
\| \mathbf{N}(u, v) \| = |v - u| \sqrt{36u^2 v^2 + 9(u^2 + 2uv + v^2) + 4}.
$$

This expression looks very impossible, so we can only hope for that some factor of the integrand cancels the unfortunate square root.

The integrand is given in the parameters of the surface by

$$
\begin{aligned}
x^3 + 2z - 3xy &= (u+v)^3 + 2(u^3 + v^3) - 3(u+v)(u^2 + v^2) \\
&= u^3 + 3u^2 v + 3uv^2 + v^3 + 2u^3 + 2v^3 - 3u^3 - 3u^2 v - 3uv^2 - 3v^3 \\
&= 0.
\end{aligned}
$$

Luckily, the surface of integration $\mathcal{F}$ is a zero surface of the integrand, so there is nothing to worry about,

$$
\int_{\mathcal{F}} (x^3 + 2z - 3xy)\, dS = 0.
$$

Example 1.5 *Let $\mathcal{F}$ be the sphere of centrum $(0,0,0)$ and radius a, and let*

$$f(x,y,z) = \alpha(x^2 + y^2 - 2z^2) + \beta xy,$$

where α and β are constants. Compute the surface integrals

$$Q = \int_{\mathcal{F}} f(x,y,z)\,dS \quad and \quad \mathbf{p} = \int_{\mathcal{F}} (x,y,z)f(x,y,z)\,dS.$$

A Surface integral.

D Exploit the symmetry of the sphere, since this is far easier than just to insert into some formula. Notice that there are several possibilities of insertion into standard formulæ, though none of them looks promising.

I It follows by the symmetry that

$$\int_{\mathcal{F}} x^2\,dS = \int_{\mathcal{F}} y^2\,dS = \int_{\mathcal{F}} z^2\,dS,$$

and that

$$\int_{\mathcal{F}} xy\,dS = 0.$$

It follows immediately that

$$Q = \alpha \left(\int_{\mathcal{F}} x^2\,dS + \int_{\mathcal{F}} y^2\,dS - 2\int_{\mathcal{F}} z^2\,dS \right) + \beta \int_{\mathcal{F}} xy\,dS = 0.$$

A similar symmetric consideration shows that if $g(x,y,z)$ is a homogeneous polynomial of odd degree, then

$$\int_{\mathcal{F}} g(x,y,z)\,dS = 0.$$

Split $\mathcal{F}$ into the eight surfaces occurring by the intersections by the three coordinate planes. By assuming that $g(x,y,z)$ is *odd*, it follows by the symmetry of the sphere that the surfaces can be paired in such a way that the sum of the surface integrals over each pair is zero. (The details are left to the reader).

Since $x\,f(x,y,z)$, $y\,f(x,y,z)$ and $z\,f(x,y,z)$ all are homogeneous of degree 3, we conclude that

$$\mathbf{p} = \mathbf{0}.$$

REMARK. We shall for obvious reasons skip the traditional variants which give a lot of tedious computations. The reason for including this example is of course to demonstrate that one in some cases may benefit from the symmetry. ◊

Example 1.6 *Let $\mathcal{F}$ be the sphere given by $r = a$ and let R denote the distance from the point (x, y, z) on the sphere to the point $(0, 0, w)$ on the Z-axis. Find*

$$U(w) = \int_{\mathcal{F}} \frac{1}{R} \, dS.$$

One may assume that $w \geq 0$. The cases $w = a$ and $w = 0$, however, must be treated separately.

A Surface integral.

D We may for symmetric reasons assume that $w \geq 0$. We shall first check where $\dfrac{1}{R}$ is harmonic. To this end we use the mean value theorem, whenever possible. Then proceed by calculating $U(w)$ directly. We get some special cases, when either $w = a$ or $w = 0$. We have an improper integral in the former case and lots of symmetry in the latter one.

I Clearly,

$$\frac{1}{R} = \frac{1}{\sqrt{x^2 + y^2 + (z - w)^2}} = \left\{ x^2 + y^2 + (z - w)^2 \right\}^{-\frac{1}{2}}.$$

It follows immediately for $w = 0$ that

$$U(0) = \int_{\mathcal{F}} \frac{1}{a} \, dS = \frac{1}{a} \operatorname{area}(\mathcal{F}) = 4\pi a.$$

REMARK. It can be mentioned aside that we get by using a so-called *Riesz transformation* that

$$U(w) = U(0) = 4\pi a \qquad \text{for } -a < w < a.$$

However, *Riesz-transformations* cannot be assumed for most readers, so we shall here give a straight proof instead. $\Diamond$

It follows from the expression of $\dfrac{1}{R}$ that $U(-w) = U(w)$, and we have again explained why we can choose $w \geq 0$.

First attempt. We first check if $\dfrac{1}{R}$ is harmonic for $(x, y, z) \neq (0, 0, w)$. We find

$$\frac{\partial}{\partial x} \left(\frac{1}{R} \right) = -x \left(\frac{1}{R} \right)^3,$$

and

$$\frac{\partial^2}{\partial x^2} \left(\frac{1}{R} \right) = -\left(\frac{1}{R} \right)^3 - 3x \left(\frac{1}{R} \right) \cdot \left\{ -x \left(\frac{1}{R} \right)^3 \right\} = -\left(\frac{1}{R} \right)^3 + 3x^2 \left(\frac{1}{R} \right)^5.$$

Then by the symmetry,

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} \left(\frac{1}{R} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{1}{R} \right) + \frac{\partial^2}{\partial z^2} \left(\frac{1}{R} \right) &= -3 \left(\frac{1}{R} \right)^3 + 3 \left(\frac{1}{R} \right)^5 \left\{ x^2 + y^2 + (z - w)^2 \right\} \\
&= -3 \left(\frac{1}{R} \right)^3 + 3 \left(\frac{1}{R} \right)^5 \cdot R^2 = 0,
\end{aligned}
$$

and the function is harmonic for $(x, y, z) \neq (0, 0, w)$.

It follows when $w > a$ from the mean value theorem that

$$U(w) = \int_{\mathcal{F}} \frac{1}{R} \, dS = \frac{1}{R(0,0,0)} \operatorname{area}(\mathcal{F}) = \frac{4\pi a^2}{w}, \qquad w > a,$$

thus in general

$$U(w) = \frac{4\pi a^2}{|w|} \qquad \text{for } |w| > a.$$

Notice that when $|w| < a$, we cannot use the argument above because of the singularity at $(0, 0, w)$ for $\dfrac{1}{R}$ which then lies inside K.

Second attempt. Split the surface $\mathcal{F}$ into an upper surface $\mathcal{F}_1$ and a lower surface $\mathcal{F}_2$. Then

$$z = \sqrt{a^2 - x^2 - y^2} \text{ på } \mathcal{F}_1, \qquad z = -\sqrt{a^2 - x^2 - y^2} \text{ på } \mathcal{F}_2.$$

The surface element is in rectangular coordinates given by

$$dS = \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dx\,dy, \qquad x^2 + y^2 < a^2,$$

and we have

$$R = \sqrt{x^2 + y^2 + (z-w)^2} = \sqrt{a^2 - (a^2 - x^2 - y^2) + (\pm\sqrt{a^2 - x^2 - y^2} - w)^2},$$

where the sign $+$ is used on $\mathcal{F}_1$, and the sign $-$ on $\mathcal{F}_2$.

Let S be the disc $\varrho^2 = x^2 + y^2 < a^2$. Then

$$U(w) = \int_{\mathcal{F}} \frac{1}{R}\, dS = \int_{\mathcal{F}_1} \frac{1}{R}\, dS + \int_{\mathcal{F}_2} \frac{1}{R}\, dS$$

$$= \int_0^{2\pi} \left\{ \int_0^a \frac{1}{\sqrt{a^2 - (a^2 - \varrho^2) + (\sqrt{a^2 - \varrho} - w)^2}} \cdot \frac{a\varrho}{\sqrt{a^2 - \varrho^2}}\, d\varrho \right\} d\varphi$$

$$+ \int_0^{2\pi} \left\{ \int_0^a \frac{1}{\sqrt{a^2 - (a^2 - \varrho^2) + (\sqrt{a^2 - \varrho} + w)^2}} \cdot \frac{a\varrho}{\sqrt{a^2 - \varrho^2}}\, d\varrho \right\} d\varphi$$

$$= 2\pi a \int_0^a \left\{ \frac{1}{\sqrt{a^2 - t^2 + (t - w)^2}} + \frac{1}{\sqrt{a^2 - t^2 + (t + w)^2}} \right\} dt$$

$$= 2\pi a \int_0^a \left\{ \frac{1}{\sqrt{a^2 + w^2 - 2tw}} + \frac{1}{\sqrt{a^2 + w^2 + 2tw}} \right\} dt$$

$$= 2\pi a \left[\frac{\sqrt{a^2 + w^2 - 2tw}}{-w} + \frac{\sqrt{a^2 + w^2 + 2tw}}{w} \right]_0^a$$

$$= \frac{2\pi a}{w} \left\{ -\sqrt{a^2 + w^2 - 2aw} + \sqrt{a^2 + w^2 + 2aw} + \sqrt{a^2} - \sqrt{a^2} \right\}$$

$$= \frac{2\pi a}{w} \{ |a + w| - |a - w| \}.$$

For $w = 0$ we get instead (cf. the above)

$$U(0) = 2\pi a \int_0^a \left\{ \frac{1}{\sqrt{a^2}} + \frac{1}{\sqrt{a^2}} \right\} dt = 2\pi a \cdot \frac{2}{a} \cdot a = 4\pi a,$$

in agreement with the previous result.

If $0 < w < a$, then

$$U(w) = \frac{2\pi a}{w} (a + w - a + w) = 4\pi a,$$

cf. the previous remark about the *Riesz transformation*.
When $w = a$, then $U(a) = 4\pi a$.
When $w > a$, then

$$U(w) = \frac{2\pi a}{w} (a + w + a - w) = \frac{4\pi a^2}{w},$$

cf. the result on harmonic functions.

Summarizing,

$$U(w) = \begin{cases} 4\pi a & \text{for } |w| \le a, \\[2ex] \dfrac{4\pi a^2}{|w|} & \text{for } |w| > a. \end{cases}$$

Example 1.7 *A surface of revolution $\mathcal{F}$ is given in semi polar coordinates (ϱ, φ, z) by*

$$z = \varrho^2, \qquad \varrho \in [0, 2], \qquad \varphi \in [0, 2\pi].$$

Sketch the meridian curve $\mathcal{M}$, and compute the surface integral

$$\int_{\mathcal{F}} \frac{1}{\sqrt{1+4z}}\, dS.$$

A Surface integral.

D Follow the guidelines.
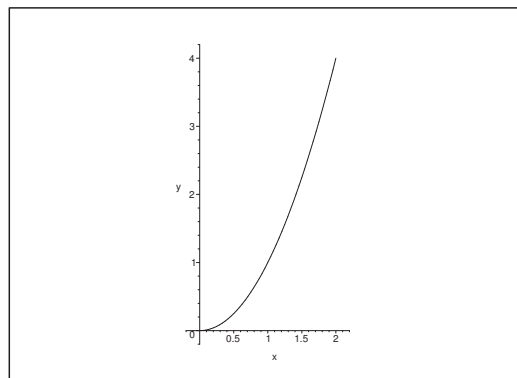
Figure 24: The meridian curve $\mathcal{M}$.

I The surface element is $dS = P\, d\varphi\, ds$, where $P = \varrho(z) = \sqrt{z}$ and

$$ds = \sqrt{1 + \left(\frac{d\varrho}{dz}\right)^2}\, dz = \sqrt{1 + \left(\frac{1}{2} \cdot \frac{1}{\sqrt{z}}\right)^2}\, dz,$$

thus

$$\begin{aligned} \int_{\mathcal{F}} \frac{1}{\sqrt{1+4z}}\, dS &= 2\pi \int_0^4 \frac{1}{1+4z} \cdot \sqrt{z} \cdot \sqrt{1 + \frac{1}{4z}}\, dz \\ &= 2\pi \int_0^4 \frac{\sqrt{z}}{1+4z} \cdot \sqrt{\frac{1+4z}{4z}}\, dz = \frac{2\pi}{2} \int_0^4 dz = 4\pi. \end{aligned}$$

Example 1.8 *A surface of revolution $\mathcal{F}$ is given in semi polar coordinates (ϱ, φ, z) by*

$$z = \varrho^3, \qquad \varrho \in \left[0, \frac{1}{2}\right], \qquad \varphi \in [0, 2\pi].$$

Sketch the meridian curve $\mathcal{M}$, and find the line element ds on this curve. Then compute the surface integral

$$\int_{\mathcal{F}} \frac{\varrho^2}{1 + 9z\varrho} \, dS.$$

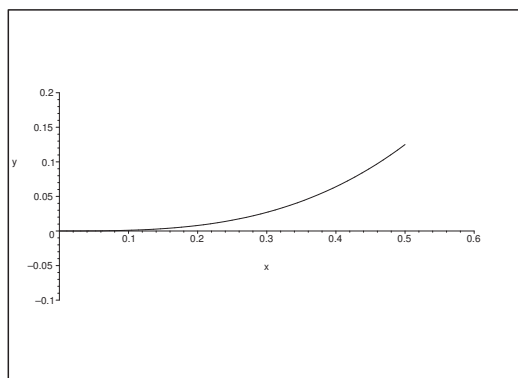**A** Surface integral.

D Follow the guidelines.

Figure 25: The meridian curve $\mathcal{M}$.

I It follows from $\dfrac{dz}{d\varrho} = 3\varrho^2$ that the line element is

$$ds = \sqrt{1 + \left(\frac{dz}{d\varrho}\right)^2} \, d\varrho = \sqrt{1 + 9\varrho^4} \, d\varrho, \qquad \varrho \in \left[0, \frac{1}{2}\right],$$

and accordingly the surface element

$$dS = \varrho \sqrt{1 + 9\varrho^4} \, d\varrho \, d\varphi, \qquad \varrho \in \left[0, \frac{1}{2}\right], \quad \varphi \in [0, 2\pi].$$

We have $z = \varrho^3$ on $\mathcal{F}$, so by insertion into the surface integralm

$$
\begin{aligned}
\int_{\mathcal{F}} \frac{\varrho^2}{1 + 9z\varrho} \, dS &= 2\pi \int_0^{\frac{1}{2}} \frac{\varrho^2}{1 + 9\varrho^4} \cdot \varrho \sqrt{1 + 9\varrho^4} \, d\varrho = 2\pi \int_0^{\frac{1}{2}} \frac{\varrho^3}{\sqrt{1 + 9\varrho^4}} \, d\varrho = \frac{2\pi}{4 \cdot 9} \int_1^{\frac{25}{16}} \frac{du}{\sqrt{u}} \\
&= \frac{2\pi}{36} [2\sqrt{u}]_1^{(5/4)^2} = \frac{4\pi}{36} \left(\frac{5}{4} - 1\right) = \frac{\pi}{36}.
\end{aligned}
$$

Example 1.9 *The surface $\mathcal{F}$ is given by*

$$z = g(x,y) = \frac{y^2}{x} + \frac{3}{4}\,x, \qquad (x,y) \in E,$$

where

$$E = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x \le 2,\, 0 \le y \le x^2\}.$$

Prove that

$$\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} = \left(\frac{y}{x}\right)^2 + \frac{5}{4},$$

and then compute the surface integral $\int_{\mathcal{F}} x\,dS$.

A Surface integral.
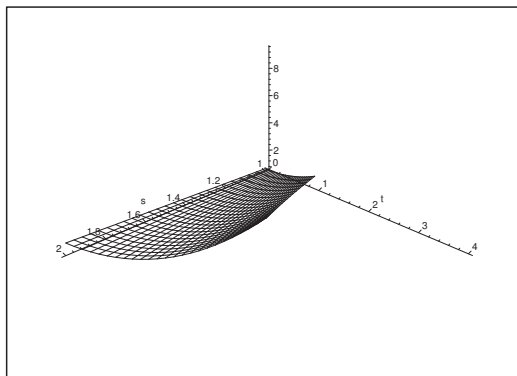
D Follow the guidelines.

Figure 26: The surface $\mathcal{F}$.

I It follows from

$$\frac{\partial g}{\partial x} = -\frac{y^2}{x^2} + \frac{3}{4}, \qquad \frac{\partial g}{\partial y} = 2\,\frac{y}{x}$$

that

$$1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 = 1 + \left\{-\frac{y^2}{x^2} + \frac{3}{4}\right\}^2 + 4 \cdot \frac{y^2}{x^2}$$

$$= 1 + \left(\frac{y}{x}\right)^4 - \frac{3}{2}\left(\frac{y}{x}\right) + \frac{9}{16} + 4\left(\frac{y}{x}\right)^2 = \left\{\left(\frac{y}{x}\right)^2\right\}^2 + \frac{5}{2}\left(\frac{y}{x}\right)^2 + \frac{25}{16}$$

$$= \left\{\left(\frac{y}{x}\right)^2\right\}^2 + 2\cdot\left(\frac{y}{x}\right)\cdot\frac{5}{4} + \left(\frac{5}{4}\right)^2 = \left\{\left(\frac{y}{x}\right)^2 + \frac{5}{4}\right\}^2,$$

hence

$$\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} = \left(\frac{y}{x}\right)^2 + \frac{5}{4}.$$

Then by the usual reduction of the surface integral to a plane integral,

$$\int_{\mathcal{F}} x\, dS = \int_E x\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dx\, dy = \int_E \left\{\left(\frac{y}{x}\right)^2 + \frac{5}{4}\right\} x\, dx\, dy$$

$$= \int_1^2 \left\{\int_0^{x^2} \left(\frac{y^2}{x} + \frac{5}{4}x\right) dy\right\} dx = \int_1^2 \left[\frac{y^3}{3x} + \frac{5}{4}xy\right]_{y=0}^{x^2} dx$$

$$= \int_1^2 \left\{\frac{x^6}{3x} + \frac{5}{4}x^3\right\} dx = \int_1^2 \left\{\frac{1}{3}x^5 + \frac{5}{4}x^3\right\} dx$$

$$= \left[\frac{1}{18}x^6 + \frac{5}{16}x^4\right]_1^2 = \frac{64}{18} + \frac{5}{16}\cdot 16 - \frac{1}{18} - \frac{5}{16} = \frac{63}{18} + \frac{75}{16} = \frac{7}{2} + \frac{75}{16} = \frac{131}{16}.$$

Example 1.10 *A plane curve* $\mathcal{L}$ *is given by the parametric description*

$$(x, y) = (\cos t, -2 \ln \sin t), \qquad t \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right].$$

1. *Show that the line element* ds *is given by*

$$ds = \frac{2 - \sin^2 t}{\sin t} \, dt.$$

A cylinder surface $\mathcal{C}$ *with* $\mathcal{L}$ *as its leading curve is given in the following way:*

$$x = \cos t, \quad y = -2 \ln \sin t, \quad z \in [0, \sin t], \quad t \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right].$$

2. *Compute the surface integral* $\int_{\mathcal{C}} xz \, dS$.

A Curve element and surface integral.

D Follow the guidelines; apply the formula of the surface integral over a cylinder surface.
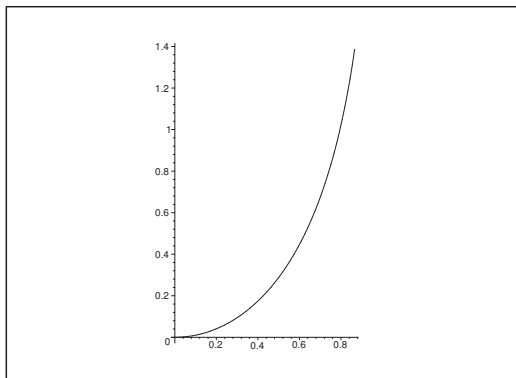
Figure 27: The leading curve $\mathcal{L}$.

I 1) From

$$\frac{dx}{dt} = -\sin t \quad \text{and} \quad \frac{dy}{dt} = -2 \frac{\cos t}{\sin t},$$

follows that

$$
\begin{aligned}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= \sin^2 t + \frac{4 \cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \left\{(\sin^2 t)^2 - 4 \sin^2 t + 4\right\} \\
&= \left\{\frac{2 - \sin^2 t}{\sin t}\right\}^2,
\end{aligned}
$$

hence

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \left|\frac{2 - \sin^2 t}{\sin t}\right| \, dt = \frac{2 - \sin^2 t}{\sin t} \, dt, \qquad t \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right].$$

Figure 28: The surface $\mathcal{F}$.

2) Then the surface integral is computed by means of the formula of an integral over a cylinder surface,

$$
\begin{aligned}
\int_{\mathcal{C}} xz \, dS &= \int_{\mathcal{L}} \left\{ \int_0^{\sin t} \cos t \cdot z \, dz \right\} ds = \int_{\mathcal{L}} \cos t \cdot \left[\frac{z^2}{2} \right]_0^{\sin t} ds \\
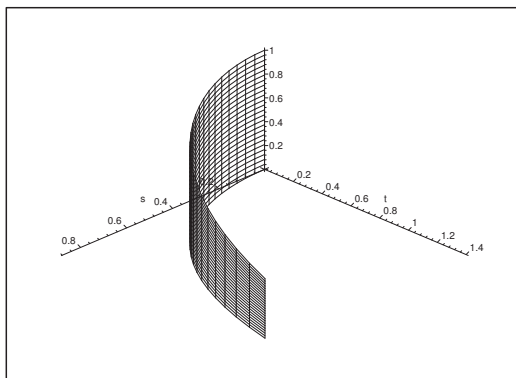&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos t \cdot \frac{1}{2} \, \sin^2 t \cdot \frac{2 - \sin^2 t}{\sin t} \, dt = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left\{ 2 \sin t - \sin^3 t \right\} \cos t \, dt \\
&= \frac{1}{2} \left[\sin^2 t - \frac{1}{4} \, \sin^4 t \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{2} \left\{ 1 - \frac{1}{4} - \frac{1}{2} + \frac{1}{4} \left(\frac{1}{2} \right)^4 \right\} \\
&= \frac{1}{128} \left\{ 64 - 16 - 32 + 1 \right\} = \frac{17}{128}.
\end{aligned}
$$

Example 1.11 *Let $\mathcal{F}$ denote the surface of the parametric description*

$$\mathbf{r}(u,v) = ((a+u)\cos v, (a+u)\sin v, av), \qquad (u,v) \in E,$$

where

$$E = \{(u,v) \in \mathbb{R}^2 \mid 0 \le u \le a, \, 0 \le v \le 2u\},$$

and where $a \in \mathbb{R}_+$ is a given constant.
Compute the surface integral

$$\int_{\mathcal{F}} \frac{z^2}{\sqrt{a^2 + x^2 + y^2}} \, dS.$$

A Surface integral.

D First find the weight function, i.e. the length of the normal vector field.

I It follows from

$$\frac{\partial \mathbf{r}}{\partial u} = (\cos v, \sin v, 0), \qquad \frac{\partial \mathbf{r}}{\partial v} = (-(a+u)\sin v, (a+u)\cos v, a),$$
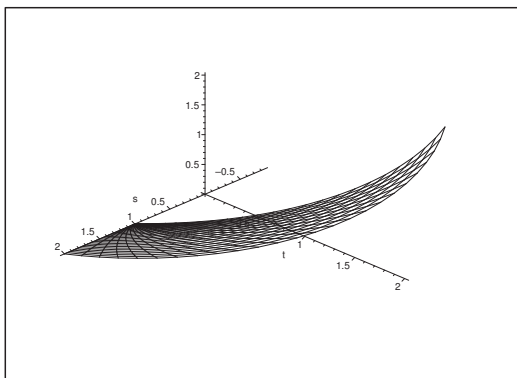
Figure 29: The surface $\mathcal{F}$ for $a = 1$.

that the normal vector is given by

$$\mathbf{N}(u,v) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \cos v & \sin v & 0 \\ -(a+u)\sin v & (a+u)\cos v & a \end{vmatrix} = (a\sin v, a\cos v, a+u),$$

thus

$$\|\mathbf{N}(u,v)\| = \sqrt{a^2 + (a+u)^2}.$$

Then we can compute the surface integral,

$$
\begin{aligned}
\int_{\mathcal{F}} \frac{z^2}{\sqrt{a^2+x^2+y^2}}\, dS &= \int_E \frac{a^2 v^2}{\sqrt{a^2+(a+u)^2}} \cdot \|\mathbf{N}(u,v)\|\, du\, dv \\
&= a^2 \int_0^a \left\{ \int_0^{2u} v^2\, dv \right\} du = a^2 \int_0^a \left[\frac{v^3}{3} \right]_0^{2u} du \\
&= \frac{8}{3} a^2 \int_0^a u^3\, du = \frac{2}{3}\, a^2 \cdot a^4 = \frac{2}{3}\, a^6.
\end{aligned}
$$

Example 1.12 *A surface of revolution $\mathcal{O}$ is given in semi polar coordinates (ϱ, φ, z) by*

$$\varrho \in [0, 2a], \qquad \varphi \in [0, 2\pi], \qquad z = \sqrt{a^2 + \varrho^2},$$

where $a \in \mathbb{R}_+$ is some given constant.

1) Sketch the meridian curve $\mathcal{M}$.

2) Show that the line element ds on $\mathcal{M}$ is given by

$$ds = \sqrt{\frac{a^2 + 2\varrho^2}{a^2 + \varrho^2}}\, d\varrho.$$

3) Compute the line integral

$$\int_{\mathcal{M}} z\varrho\, ds.$$

4) Compute the surface integral

$$\int_{\mathcal{O}} \frac{1}{z^2 \sqrt{z^2 + \varrho^2}}\, dS.$$

A Surface of revolution, line integral and surface integral.

D Standard example.

2) From

$$dz = \frac{\varrho}{\sqrt{a^2 + \varrho^2}}\, d\varrho,$$

follows that

$$ds = \sqrt{(d\varrho)^2 + (dz)^2} = \sqrt{1 + \frac{\varrho^2}{a^2 + \varrho^2}}\, d\varrho = \sqrt{\frac{a^2 + 2\varrho^2}{a^2 + \varrho^2}}\, d\varrho.$$

I 1)
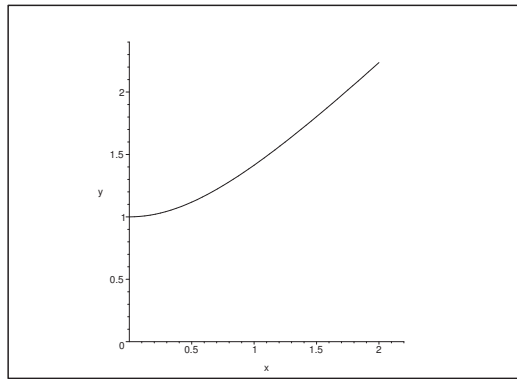
Figure 30: The meridian curve $\mathcal{M}$ for $a = 1$.

3) We conclude from 2) that

$$
\begin{aligned}
\int_{\mathcal{M}} z\varrho\, ds &= \int_0^{2a} \sqrt{a^2 + \varrho^2} \cdot \varrho \cdot \sqrt{\frac{a^2 + 2\varrho^2}{a^2 + \varrho^2}}\, d\varrho = \int_0^{2a} \sqrt{a^2 + 2\varrho^2} \cdot \varrho\, d\varrho \\
&= \frac{1}{4} \cdot \frac{2}{3} \left[\left(a^2 + 2\varrho^2\right)^{\frac{3}{2}} \right]_{\varrho=0}^{2a} = \frac{1}{6} \left\{ \left(9a^2\right)^{\frac{3}{2}} \right\} = \frac{1}{6}\, a^3 \left(3^3 - 1\right) = \frac{26}{6}\, a^3 = \frac{13}{3}\, a^3.
\end{aligned}
$$

4) Again we get by first applying the result of 2),

$$
\int_{\mathcal{O}} \frac{1}{z^2 \sqrt{z^2 + \varrho^2}}\, dS = 2\pi \int_0^{2\pi} \frac{1}{(a^2 + \varrho^2)\sqrt{a^2 + 2\varrho^2}} \cdot \varrho \sqrt{\frac{a^2 + 2\varrho^2}{a^2 + \varrho^2}}\, d\varrho
$$

$$
= 2\pi \int_0^{2a} \left(a^2 + \varrho^2\right)^{-\frac{3}{2}} \varrho\, d\varrho = 2\pi \left[-\frac{1}{\sqrt{a^2 + \varrho^2}} \right]_{\varrho=0}^{2a}
$$

$$
= 2\pi \left(\frac{1}{a} - \frac{1}{\sqrt{5}} \cdot \frac{1}{a} \right) = \frac{2\pi}{a}\left(1 - \frac{1}{\sqrt{5}} \right) = \frac{2(5 - \sqrt{5})\pi}{5a}.
$$

Example 1.13 *A surface of revolution $\mathcal{O}$ is given in semi polar coordinates by*

$$
\varrho \in [a, 2a], \qquad \varphi \in [0, 2\pi], \qquad z = 2a - \frac{\varrho^2}{a},
$$

where $a \in \mathbb{R}_+$ is some given constant.

1) Sketch the meridian curve $\mathcal{M}$, and show that the line element ds on $\mathcal{M}$ is given by

$$
ds = \frac{1}{a}\sqrt{a^2 + 4\varrho^2}\, d\varrho.
$$

2) Compute the line integral

$$
\int_{\mathcal{M}} \sqrt{2 - \frac{z}{a}}\, ds.
$$

3) Compute the surface integral

$$
\int_{\mathcal{O}} \frac{1}{az + 9\varrho^2}\, dS.
$$

A Line integral and surface integral.

D Apply the standard methods.

I 1) When we use the parametric description

$$
\mathcal{M}: \quad (\varrho, z) = \left(\varrho, 2a - \frac{\varrho^2}{a} \right), \qquad \varrho \in [a, 2a],
$$

the square of the weight function becomes

$$
\left(\frac{d\varrho}{d\varrho} \right)^2 + \left(\frac{dz}{d\varrho} \right)^2 = 1 + \left(-\frac{2\varrho}{a} \right)^2 = \frac{1}{a^2}\left(a^2 + 4\varrho^2 \right),
$$

hence

$$
ds = \frac{1}{a}\sqrt{a^2 + 4\varrho^2}\, d\varrho.
$$
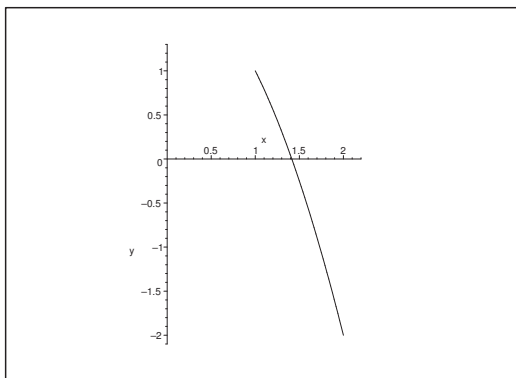
Figure 31: The meridian curve $\mathcal{M}$ for $a = 1$.

2) Then by 1) and the substitution $t = 4\varrho^2$,

$$
\begin{aligned}
\int_{\mathcal{M}} \sqrt{2 - \frac{z}{a}}\, ds &= \int_a^{2a} \sqrt{2 - \left(2 - \frac{\varrho^2}{a^2}\right)} \cdot \frac{1}{a}\, \sqrt{a^2 + 4\varrho^2}\, d\varrho \\
&= \int_a^{2a} \frac{\varrho}{a^2}\, \sqrt{a^2 + 4\varrho^2}\, d\varrho = \frac{1}{8a^2} \int_{4a^2}^{16a^2} \sqrt{a^2 + t}\, dt \\
&= \frac{1}{8a^2} \left[\frac{2}{3}\left(a^2 + t\right)^{\frac{3}{2}}\right]_{4a^2}^{16a^2} = \frac{1}{12a^2} \left\{(17a^2)^{\frac{3}{2}} - (5a^2)^{\frac{3}{2}}\right\} \\
&= \frac{17\sqrt{17} - 5\sqrt{5}}{12}\, a.
\end{aligned}
$$

3) By first intersecting the surface $\mathcal{O}$ with the planes $z = $ constant, we get

$$
\begin{aligned}
\int_{\mathcal{O}} \frac{1}{az + 9\varrho^2}\, dS &= \int_{\mathcal{M}} \frac{2\pi\varrho}{a\left(2a - \dfrac{\varrho^2}{a}\right) + 9\varrho^2}\, ds = \int_a^{2a} \frac{2\pi\varrho}{2a^2 + 8\varrho^2} \cdot \frac{1}{a}\, \sqrt{a^2 + 4\varrho^2}\, d\varrho \\
&= \frac{\pi}{a} \int_a^{2a} \frac{\varrho}{\sqrt{a^2 + 4\varrho^2}}\, d\varrho = \frac{\pi}{a} \left[\frac{1}{4}\, \sqrt{a^2 + 4\varrho^2}\right]_a^{2a} = \frac{\pi}{4a} \left\{\sqrt{17a^2} - \sqrt{5a^2}\right\} \\
&= \frac{\pi}{4}\left(\sqrt{17} - \sqrt{5}\right).
\end{aligned}
$$

Example 1.14 *A surface of revolution $\mathcal{O}$ is given in semi polar coordinates (ϱ, φ, z) by*

$$\varrho = \sqrt{z^2 + 2az}, \qquad z \in [a, 2a], \qquad \varphi \in [0, 2\pi],$$

where a is some positive constant. The meridian curve of the surface is called $\mathcal{M}$.

1) Explain why $\mathcal{M}$ is a subset of a conic section, and indicate its type and centrum. Then sketch $\mathcal{M}$.

2) Show that the line element ds on $\mathcal{M}$ is given by

$$ds = \sqrt{\frac{2z^2 + 4az + a^2}{z^2 + 2az}} \, dz.$$

3) Compute the surface integral

$$\int_{\mathcal{O}} \frac{|x|(z+a)}{\sqrt{x^2 + y^2}} \, dS.$$

4) Explain why $\mathcal{O}$ is a subset of a surface of a conic section. Find its type and centrum.

A Conic sections, meridian curve, surface integral.

D If only the surface integral is calculated in semi polar coordinates, the rest is purely standard.
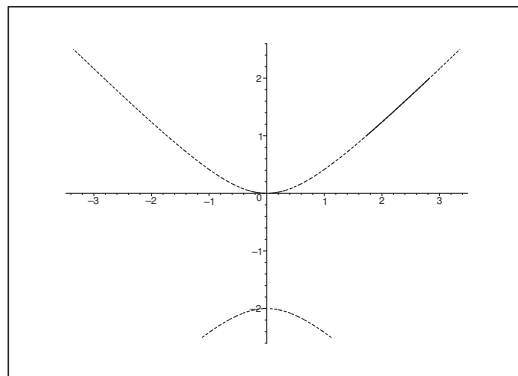
Figure 32: The meridian curve $\mathcal{M}$ and the corresponding conic section (dotted) for $a = 1$.

I 1) We get by a squaring and a rearrangement that $\mathcal{M}$ is a subset of the point set given by

$$(z + a)^2 - \varrho^2 = a^2.$$

This describes in the whole PZ-plane an hyperbola of centrum $(0, -a)$ and half axes a and a.

2) The line element on $\mathcal{M}$ is given by

$$
\begin{aligned}
ds &= \sqrt{1 + \left(\frac{d\varrho}{dz}\right)^2} \, dz = \sqrt{1 + \left(\frac{2z + 2a}{2\sqrt{z^2 + 2az}}\right)^2} \, dz \\
&= \sqrt{1 + \frac{z^2 + 2az + a^2}{z^2 + 2az}} \, dz = \sqrt{\frac{2z^2 + 4az + a^2}{z^2 + 2az}} \, dz.
\end{aligned}
$$

3) First express the integrand in semi polar coordinates on the surface:

$$f(x, y, z) = \frac{|x|(z+a)}{\sqrt{x^2 + y^2}} = \frac{\varrho |\cos \varphi|\,(z+a)}{\varrho} = |\cos \varphi|(z+a).$$

Then the surface integral becomes

$$
\begin{aligned}
\int_{\mathcal{O}} \frac{|x|(z+a)}{\sqrt{x^2+y^2}}\, dS &= \int_a^{2a} \left\{ \int_0^{2\pi} |\cos\varphi|(z+a)\varrho(z)\, d\varphi \right\} \sqrt{\frac{2z^2+4az+a^2}{z^2+2az}}\, dz \\
&= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\, d\varphi \cdot \int_a^{2a} (z+a)\sqrt{z^2+2az} \cdot \sqrt{\frac{2z^2+4aza^2}{z^2+2az}}\, dz \\
&= 4 \int_a^{2a} (z+a)\sqrt{2z^2+4az+a^2}\, dz \\
&= \int_{z=a}^{2a} \sqrt{2z^2+4az+a^2}\, d\left(2z^2+4az+a^2\right) \\
&= \frac{2}{3} \left[\left(2z^2+4az+a^2\right)^{\frac{3}{2}} \right]_{z=a}^{2a} \\
&= \frac{2}{3} \left\{ \left(8a^2+8a^2+a^2\right)^{\frac{3}{2}} - \left(2a^2+4a^2+a^2\right)^{\frac{3}{2}} \right\} \\
&= \frac{2}{3} \left\{ \left(17a^2\right)^{\frac{3}{2}} - \left(7a^2\right)^{\frac{3}{2}} \right\} = \frac{2}{3}\{17\sqrt{17} - 7\sqrt{7}\}a^3.
\end{aligned}
$$

4) The curve $\mathcal{M}$ is a part of an hyperbola, cf. 1), and the axis of rotation intersects the foci of the hyperbola. We therefore conclude that $\mathcal{O}$ is a subset of an hyperboloid of revolution with two nets and centrum $(0,0,-a)$.

We get the equation of the hyperboloid of revolution by replacing ϱ^2 by x^2+y^2 in the expression from 1),

$$(z+a)^2 - x^2 - y^2 = a^2,$$

or in its standard form,

$$\left(\frac{z+a}{a}\right)^2 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2 = 1.$$

The surface $\mathcal{O}$ it that subset which lies between the planes $z = a$ and $z = 2a$.

Example 1.15 *A surface of revolution $\mathcal{O}$ is given in semi polar coordinates (ϱ, φ, z) by*

$$0 \leq \varphi \leq 2\pi, \qquad a \leq \varrho \leq 2a, \qquad z = a \ln \frac{\varrho}{a},$$

where a is some positive constant.

1) Sketch the meridian curve $\mathcal{M}$, and find the line element ds on $\mathcal{M}$.

2) Compute the line integral

$$\int_{\mathcal{M}} \frac{1}{\sqrt{a^2 + \varrho^2}}\, ds.$$

3) Compute the surface integral

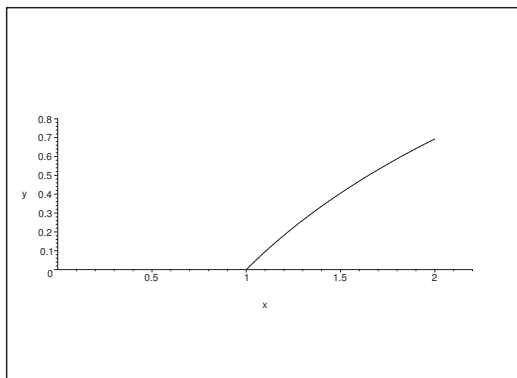$$\int_{\mathcal{O}} \left(x + a \exp \frac{z}{a}\right) dS.$$

Figure 33: The meridian curve $\mathcal{M}$ for $a = 1$.

A Surface of revolution, meridian curve, line integral, surface integral.

D Standard example.

I 1) The line element ds on $\mathcal{M}$ is given by

$$ds = \sqrt{1 + \left(\frac{dz}{d\varrho}\right)^2}\, d\varrho = \sqrt{1 + \left(\frac{a}{\varrho}\right)^2}\, d\varrho = \frac{\sqrt{a^2 + \varrho^2}}{\varrho}\, d\varrho.$$

2) By using ϱ as variable it follows from 1) that

$$\int_{\mathcal{M}} \frac{1}{\sqrt{a^2 + \varrho^2}}\, ds = \int_a^{2a} \frac{1}{\sqrt{a^2 + \varrho^2}} \cdot \frac{\sqrt{a^2 + \varrho^2}}{\varrho}\, d\varrho = \int_a^{2a} \frac{d\varrho}{\varrho} = [\ln \varrho]_a^{2a} = \ln 2.$$

3) The surface element on $\mathcal{O}$ is given by

$$dS = \varrho\, d\varphi\, ds = \varrho\, \frac{\sqrt{a^2 + \varrho^2}}{\varrho}\, d\varphi\, d\varrho = \sqrt{a^2 + \varrho^2}\, d\varrho\, d\varphi,$$

so accordingly the surface integral

$$
\begin{aligned}
\int_{\mathcal{O}} \left\{ x + a \exp\left(\frac{z}{a}\right) \right\} dS &= \int_0^{2\pi} \left\{ \int_a^{2a} \left(\varrho \cos\varphi + a \exp\left(\frac{a \ln \frac{\varrho}{a}}{a}\right) \right) \sqrt{a^2 + \varrho^2}\, d\varrho \right\} d\varphi \\
&= 0 + 2\pi \int_a^{2a} \varrho \sqrt{a^2 + \varrho^2}\, d\varrho = \pi \int_{\varrho=a}^{2a} \left(a^2 + \varrho^2\right)^{\frac{1}{2}}\, d\left(a^2 + \varrho^2\right) \\
&= \pi \cdot \frac{2}{3} \left[\left(a^2 + \varrho^2\right)^{\frac{3}{2}} \right]_{\varrho=a}^{2a} = \frac{2\pi}{3} \left\{ \left(5a^2\right)^{\frac{3}{2}} - \left(2a^2\right)^{\frac{3}{2}} \right\} \\
&= \frac{2\pi}{3} \left(5\sqrt{5} - 2\sqrt{2}\right) a^3.
\end{aligned}
$$

Example 1.16 *A surface $\mathcal{F}$ is given by the parametric description*

$$\mathbf{r}(u,v) = (e^u, e^v, u+v), \qquad u^2 + v^2 \leq 1.$$

1) Show that the normal vector of the surface is given by

$$\mathbf{N}(u,v) = \left(-e^x, -e^u, e^{u+v}\right).$$

2) Find an equation of the tangent plane of $\mathcal{F}$ at the point $\mathbf{r}(0,0)$.

3) Compute the surface integral

$$\int_{\mathcal{F}} \frac{1}{\sqrt{x^2 + y^2 + e^{2z}}}\, dS.$$

A Surface integral.

D Use that $dS = \|\mathbf{N}(u,v)\|\, du\, dv$.
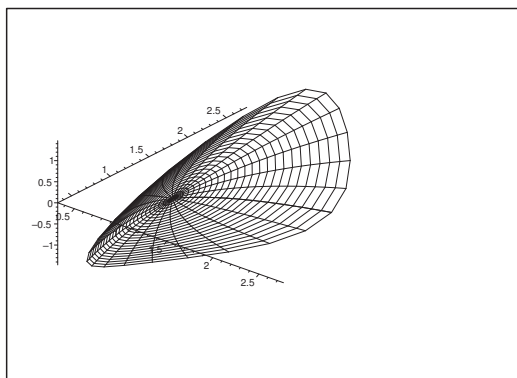
Figure 34: The surface $\mathcal{F}$.

I 1) We conclude from

$$\frac{\partial \mathbf{r}}{\partial u} = (e^u, 0, 1) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} = (0, e^v, 1),$$

that

$$\mathbf{N}(u,v) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ e^u & 0 & 1 \\ 0 & e^v & 1 \end{vmatrix} = \left(-e^v, -e^u, e^{u+v}\right).$$

2) From $\mathbf{r}(0,0) = (1,1,0)$ and the normal vector $\mathbf{N}(0,0) = (-1,-1,1)$ we get the equation of the tangent plane

$$0 = \mathbf{N}(0,0,0) \cdot (x-1, y-1, z) = (-1,-1,1) \cdot (x-1, y-1, z) = -x + 1 - y + 1 + z,$$

thus by a rearrangement

$$x + y - z = 2.$$

3) From $\|\mathbf{N}(u,v)\|^2 = e^{2v} + e^{2u} + e^{2u+2v}$ follows that

$$\int_{\mathcal{F}} \frac{1}{\sqrt{x^2 + y^2 + e^{2z}}} \, dS = \int_{u^2+v^2 \leq 1} \frac{\sqrt{e^{2u} + e^{2v} + e^{2u+2v}}}{\sqrt{e^{2u} + e^{2v} + e^{2u+2v}}} \, du \, dv = \pi \cdot 1^2 = \pi.$$

I

Figure 35: The surface of **Example 2.1.1**.

2 Surface area

Example 2.1 *Compute in each of the following cases the surface area of a surface $\mathcal{F}$, which is the graph of a function in two variables, thus*
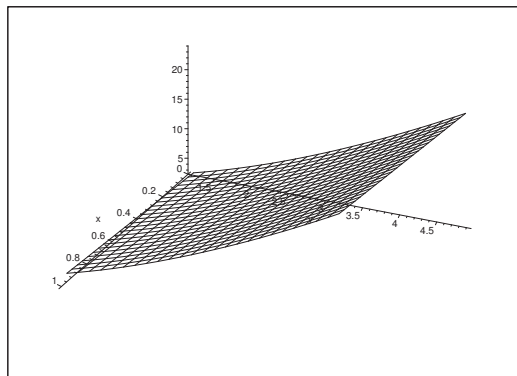
$$\mathcal{F} = \{(x, y, z) \mid (x, y) \in E,\ z = Z(x, y)\}.$$

1) The surface integral $\int_{\mathcal{F}} dS$, where

$$Z(x, y) = 1 + 2x + 2y\sqrt{y}, \qquad (x, y) \in [0, 1] \times \left[\frac{11}{9}, \frac{44}{9}\right].$$

2) The surface integral $\int_{\mathcal{F}} dS$, where

$$Z(x, y) = \frac{x^2}{2} + 3y, \qquad hvor \ -1 \le x \le 1 \ og \ -\frac{1}{6}x^2 \le y \le 1.$$

3) The surface integral $\int_{\mathcal{F}} dS$, where

$$Z(x, y) = \frac{y}{x^2 + y^2}, hvor\ 1 \le x^2 + y^2 \le 2.$$

A Surface area in rectangular coordinates.

D Find the weight function

$$\|\mathbf{N}\| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} = \sqrt{1 + \| \bigtriangledown g \|^2},$$

and then compute the surface integral with the integrand 1.

Here $\bigtriangledown g = (2, 3\sqrt{y})$, so the weight function is

$$\sqrt{1 + \| \bigtriangledown g \|^2} = \sqrt{1 + 4 + 9y} = \sqrt{5 + 9y},$$

1)
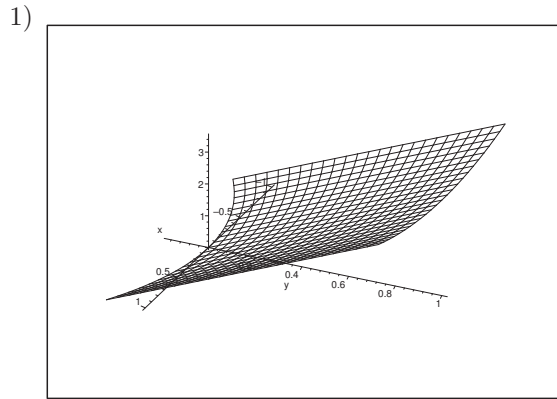
Figure 36: The surface of **Example 2.1.2**.

and we find the surface integral

$$
\int_{\mathcal{F}} dS \;=\; \int_{E} \sqrt{5+9y}\,dx\,dy = \int_{\frac{11}{9}}^{\frac{44}{9}} \sqrt{5+9y}\,dy = \frac{1}{9}\cdot\frac{2}{3}\left[(5+9y)^{\frac{3}{2}}\right]_{\frac{11}{9}}^{\frac{44}{9}}
$$

$$
=\; \frac{2}{27}\left\{49^{\frac{3}{2}}-16^{\frac{3}{2}}\right\}\Big)\frac{2}{27}\left\{7^{3}-4^{3}\right\}=\frac{2}{27}(343-64)=\frac{2}{27}\cdot 279=\frac{2\cdot 31}{3}=\frac{62}{3}.
$$
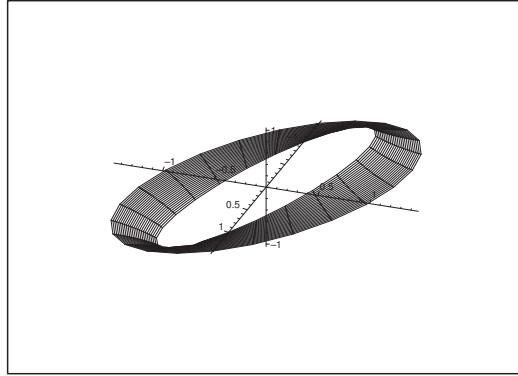
Figure 37: The surface of **Example 2.1.3**.

2) We get from $\bigtriangledown g = (x, 3)$ that $\sqrt{1 + \| \bigtriangledown g \|^2} = \sqrt{10 + x^2}$. The surface area is

$$
\begin{aligned}
\int_{\mathcal{F}} dS &= \int_E \sqrt{10 + x^2} \, dx \, dy = \int_{-1}^1 \left\{ \int_{-\frac{x^2}{6}}^1 \sqrt{10 + x^2} \, dy \right\} dx \\
&= \int_{-1}^1 \left(1 + \frac{x^2}{6} \right) \sqrt{10 + x^2} \, dx = \frac{2}{6} \int_0^1 (6 + x^2) \sqrt{10 + x^2} \, dx.
\end{aligned}
$$

Then by the substitution $x = \sqrt{10} \sinh t$, $t = \text{Arsinh}\left(\dfrac{x}{\sqrt{10}} \right)$,

$$
\begin{aligned}
\int_{\mathcal{F}} dS &= \frac{1}{3} \int_0^{\text{Arsinh}(\frac{1}{\sqrt{10}})} (6 + 10 \sinh^2 t) \cdot \sqrt{10} \cosh t \cdot \sqrt{10} \cosh t \, dt \\
&= \frac{20}{3} \int_0^{\text{Arsinh}(\frac{1}{\sqrt{10}})} (3 + 5 \sinh^2 t) \cosh^2 t \, dt \\
&= \frac{20}{3} \int_0^{\text{Arsinh}(\frac{1}{\sqrt{10}})} \left\{ \frac{3}{2}(1 + \cosh 2t) + \frac{5}{4} \sinh^2 2t \right\} dt \\
&= \frac{5}{3} \int_0^{\text{Arsinh}(\frac{1}{\sqrt{10}})} \left\{ 6 + 6 \cosh 2t + \frac{5}{2}(\cosh 4t - 1) \right\} dt \\
&= \frac{5}{6} \int_0^{\text{Arsinh}(\frac{1}{\sqrt{10}})} \{ 7 + 12 \cosh 2t + 5 \cosh 4t \} dt \\
&= \frac{5}{6} \left[7t + 6 \sinh 2t + \frac{5}{4} \sinh 4t \right]_0^{\text{Arsinh}(\frac{1}{\sqrt{10}})} \\
&= \frac{5}{6} \left[7t + 12 \sinh t \sqrt{1 + \sinh^2 t} + 5 \sinh t \sqrt{1 + \sinh^2 t} \cdot (1 + 2 \sinh^2 t) \right]_0^{\text{Arsinh}(\frac{1}{\sqrt{10}})} \\
&= \frac{5}{6} \left\{ 7 \ln \left(\frac{1}{\sqrt{10}} + \sqrt{\frac{11}{10}} \right) + 12 \cdot \frac{1}{\sqrt{10}} \cdot \sqrt{\frac{11}{10}} + 5 \cdot \frac{1}{\sqrt{10}} \cdot \sqrt{\frac{11}{10}} \cdot \left(1 + \frac{2}{10} \right) \right\} \\
&= \frac{5}{6} \left\{ 7 \ln \left(\frac{1 + \sqrt{11}}{\sqrt{10}} \right) + \frac{12}{10} \cdot \sqrt{11} + \frac{6}{10} \cdot \sqrt{11} \right\} = \frac{35}{12} \ln \left(\frac{6 + \sqrt{11}}{5} \right) + \frac{3}{2} \cdot \sqrt{11}.
\end{aligned}
$$

3) Here

$$\bigtriangledown g = \left(-\frac{2xy}{(x^2+y^2)^2}, \frac{x^2-y^2}{(x^2+y^2)^2}\right) = \frac{1}{(x^2+y^2)^2}\left(-2xy, x^2-y^2\right),$$

so

$$\|\bigtriangledown g\|^2 = \frac{1}{(x^2+y^2)^4}\left(4x^2y^2 + (x^2-y^2)^2\right) = \frac{1}{(x^2+y^2)^2}.$$

The surface area is

$$
\begin{aligned}
\int_{\mathcal{F}} dS &= \int_E \frac{1}{x^2+y^2}\sqrt{1+(x^2+y^2)^2}\, dx\, dy = 2\pi \int_0^{\sqrt{2}} \frac{\sqrt{1+\varrho^4}}{\varrho^2}\, \varrho\, d\varrho \\
&= \frac{2\pi}{4}\int_1^{\sqrt{2}} \frac{\sqrt{1+\varrho^4}}{\varrho^4}\cdot 4\varrho^3\, d\varrho = \frac{\pi}{2}\int_1^4 \frac{1+t}{t}\, dt = \frac{\pi}{2}\int_{\sqrt{2}}^{\sqrt{5}} \frac{u\cdot 2u}{u^2-1}\, du \\
&= \pi\int_{\sqrt{2}}^{\sqrt{5}}\left\{1 + \frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}\right\} du = \pi\left[u + \frac{1}{2}\ln\frac{u-1}{u+1}\right]_{\sqrt{2}}^{\sqrt{5}} \\
&= \pi\left\{\sqrt{5}-\sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right\} \\
&= \pi\left\{\sqrt{5}-\sqrt{2} + \ln\left(\frac{(\sqrt{5}-1)(\sqrt{2}+1)}{2}\right)\right\} \\
&= \pi\{\sqrt{5}-\sqrt{2} + \ln(\sqrt{5}-1) + \ln(\sqrt{2}+1) - \ln 2\}.
\end{aligned}
$$

Example 2.2 *Calculate in each of the following cases the surface area of a surface of revolution $\mathcal{O}$, which is given by a meridian curve $\mathcal{M}$ in the meridian half plane, in which ϱ and z are the rectangular coordinates.*

1) The surface area $\int_{\mathcal{O}} dS$, where the meridian curve $\mathcal{M}$ is given by the parametric description

$$(\varrho, z) = \left(2\sin^3 t, 3\cos t - 2\cos^3 t\right), \qquad t \in \left[0, \frac{\pi}{2}\right].$$

2) The surface area $\int_{\mathcal{O}} dS$, where the meridian curve $\mathcal{N}$ is given by the parametric description

$$(\varrho, z) = \left(a\sin^3 t, a\cos^3 t\right), \qquad t \in [0, \pi].$$

3) The surface area $\int_{\mathcal{O}} dS$, where the meridian curve $\mathcal{M}$ is given by the parametric description

$$(\varrho, z) = (b\sin t, a\cos t), \qquad t \in [0, \pi].$$

4) The surface area $\int_{\mathcal{O}} dS$, where the meridian curve $\mathcal{M}$ is given by $z^2 + \varrho^2 = az$.

5) The surface area $\int_{\mathcal{O}} dS$, where the meridian curve $\mathcal{M}$ is given by $\varrho = z^3$ for $x \in [0,1]$.

I
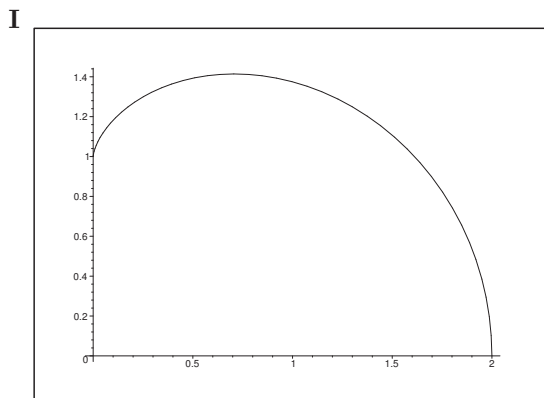
Figure 38: The meridian curve $\mathcal{M}$ of **Example 2.2.1**.

A Surface area of a surface of revolution.

D Use either semi polar or spherical coordinates and the area element $\varrho\,d\varphi\,ds$, where ds is the line element, thus if e.g. $z = g(\varrho)$, then

$$ds = \sqrt{1 + g'(\varrho)^2}\,d\varrho,$$

and similarly.

We get from

$$\mathbf{r}(t) = \left(2\sin^3 t,\, 3\cos t - 2\cos^3 t\right)$$

that

$$\mathbf{r}'(t) = \left(6\sin^2 t \cdot \cos t,\, -3\sin t + 6\cos^2 t \cdot \sin t\right),$$

thus

$$
\begin{aligned}
\|\mathbf{r}'(t)\|^2 &= \left(6\sin^2 t \cdot \cos t\right)^2 + \left(-3\sin t + 6\cos^2 t \cdot \sin t\right)^2 \\
&= 36\sin^4 t \cdot \cos^2 + 9\sin^2 t \left(2\cos^2 t - 1\right)^2 \\
&= 9\sin^2 t \left(\sin^2 2t + \cos^2 2t\right) = 9\sin^2 t,
\end{aligned}
$$

and accordingly

$$ds = \|\mathbf{r}'(t)\|\,dt = 3|\sin t|\,dt = 3\sin t\,dt \quad \text{for } t \in \left[0, \frac{\pi}{2}\right].$$

Then

$$
\begin{aligned}
\int_{\mathcal{O}} dS &= \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}} 2\sin^3 t \cdot 3\sin t\,dt \right\} d\varphi = 2\pi \cdot 6 \int_0^{\frac{\pi}{2}} \sin^4 t\,dt \\
&= 3\pi \int_0^{\frac{\pi}{2}} \left\{2\sin^2 t\right\}^2 dt = 3\pi \int_0^{\frac{\pi}{2}} (1 - \cos 2t)^2\,dt \\
&= 3\pi \int_0^{\frac{\pi}{2}} \left\{ 1 - 2\cos 2t + \frac{1}{2} + \frac{1}{2}\cos 4t \right\} dt = 3\pi \cdot \frac{3}{2} \cdot \frac{\pi}{2} = \frac{9\pi^2}{4}.
\end{aligned}
$$

1)
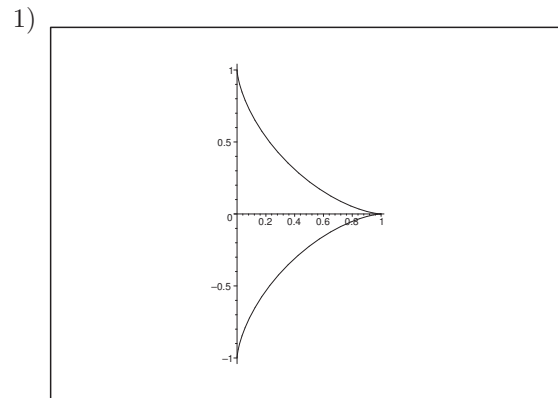
Figure 39: The meridian curve $\mathcal{M}$ of **Example 2.2.2** for $a = 1$.
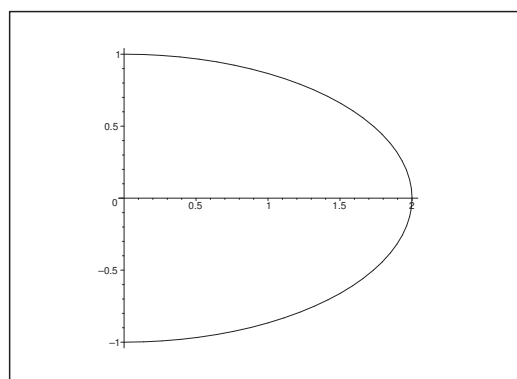
Figure 40: The meridian curve $\mathcal{M}$ of **Example 2.2.3** for $a = 1$ og $b = 2$.

2) From $\mathbf{r}(t) = a\left(\sin^3 t, \cos^3 t\right)$ follows that

$$\mathbf{r}'(t) = a\left(3\sin^2 t \,\cos t, -3\cos^2 t \,\sin t\right) = 3a\sin t \cdot \cos t(\sin t, -\cos t),$$

hence

$$\|\mathbf{r}'(t)\| = 3a\sin t \cdot |\cos t|, \qquad t \in [0, \pi].$$

(Remember the absolute value). The line element is

$$ds = \|\mathbf{r}'(t)\|\, dt = 3a\sin t\, |\cos t|\, dt.$$

Finally, it follows from $\varrho\, d\varphi = a\sin^3 t\, d\varphi$ that

$$
\begin{aligned}
\int_{\mathcal{O}} dS &= \int_0^{2\pi} \left\{ \int_0^{\pi} a\sin^3 t \cdot 3a\sin t\, |\cos t|\, dt \right\} d\varphi \\
&= 2\pi \cdot 3a^2 \cdot 2 \int_0^{\frac{\pi}{2}} \sin^4 t \,\cos t\, dt = \frac{2\pi}{5}\, a^2.
\end{aligned}
$$

3) Here

$$ds = \|\mathbf{r}'(t)\|\, dt = \sqrt{b^2 \cos^2 t + a^2 \sin^2 t}\, dt = \sqrt{a^2 + (b^2 - a^2)\cos^2 t}\, dt, \qquad t \in [0, \pi],$$

thus

$$\int_{\mathcal{O}} dS = \int_0^{2\pi} \left\{ \int_0^{\pi} b \sin t \sqrt{a^2 + (b^2 - a^2)\cos^2 t}\, dt \right\} d\varphi = 4\pi b \int_0^1 \sqrt{a^2 + (b^2 - a^2)u^2}\, du.$$

We shall here consider three different cases.

a) If $a = b$, then

$$\int_{\mathcal{O}} dS = 4\pi a \int_0^1 a\, du = 4\pi a^2,$$

and the surface area of the sphere is $4\pi a^2$.

b) If $0 < b < a$, then

$$\int_{\mathcal{O}} dS = 4\pi ba \int_0^1 \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)u^2}\, du.$$

Download free books at BookBooN.com

Then by the substitution $\sqrt{1 - \dfrac{b^2}{a^2}}\, u = \sin v$,

$$
\begin{aligned}
\int_{\mathcal{O}} dS &= 4\pi ab \int_0^{\operatorname{Arcsin}\left(\sqrt{1-\frac{b^2}{a^2}}\right)} \sqrt{1 - \sin^2 v} \cdot \frac{1}{\sqrt{1 - \dfrac{b^2}{a^2}}} \cos v\, dv \\[2ex]
&= \frac{4\pi ab}{\sqrt{1 - \dfrac{b^2}{a^2}}} \int_0^{\operatorname{Arccos}\left(\frac{b}{a}\right)} \cos^2 v\, dv = \frac{2\pi ab}{\sqrt{1 - \dfrac{b^2}{a^2}}} \int_0^{\operatorname{Arccos}\left(\frac{b}{a}\right)} (1 + \cos 2v)\, dv \\[2ex]
&= \frac{2\pi ab}{\sqrt{1 - \dfrac{b^2}{a^2}}} \left\{ \operatorname{Arccos}\left(\frac{b}{a}\right) + \left[\frac{1}{2} \sin 2v\right]_0^{\operatorname{Arccos}\left(\frac{b}{a}\right)} \right\} \\[2ex]
&= \frac{2\pi ab}{\sqrt{1 - \dfrac{b^2}{a^2}}} \left\{ \operatorname{Arccos}\left(\frac{b}{a}\right) + \sqrt{1 - \dfrac{b^2}{a^2}} \cdot \frac{b}{a} \right\} = \frac{2\pi ab}{\sqrt{1 - \dfrac{b^2}{a^2}}}\ \operatorname{Arccos}\left(\frac{b}{a}\right) + 2\pi b^2.
\end{aligned}
$$

c) If $0 < a < b$, then

$$
\int_{\mathcal{O}} dS = 4\pi ab \int_0^1 \sqrt{1 + \left(\frac{b^2}{a^2} - 1\right) u^2}\, du.
$$

Then by the substitution $\sqrt{\dfrac{b^2}{a^2} - 1}\, u = \sinh v$,

$$
\begin{aligned}
\int_{\mathcal{O}} dS &= 4\pi ab \int_0^{\operatorname{Arsinh}\left(\sqrt{\frac{b^2}{a^2}-1}\right)} \sqrt{1 + \sinh^2 v} \cdot \frac{1}{\sqrt{\dfrac{b^2}{a^2} - 1}} \cosh v\, dv \\[2ex]
&= \frac{4\pi ab}{\sqrt{\dfrac{b^2}{a^2}}} \int_0^{\ln\left(\frac{b}{a} + \sqrt{\frac{b^2}{a^2}-1}\right)} \cosh^2 v\, dv = \frac{2\pi ab}{\sqrt{\dfrac{b^2}{a^2}}} \int_0^{\ln\left(\frac{b}{a} + \sqrt{\frac{b^2}{a^2}-1}\right)} (\cosh 2v + 1)\, dv \\[2ex]
&= \frac{2\pi ab}{\sqrt{\dfrac{b^2}{a^2} - 1}} \left\{ \ln\left(\frac{b}{a} + \sqrt{\dfrac{b^2}{a^2} - 1}\right) + \sqrt{\dfrac{b^2}{a^2} - 1} \cdot \frac{b}{a} \right\} \\[2ex]
&= \frac{2\pi ab}{\sqrt{\dfrac{b^2}{a^2} - 1}} \ln\left(\frac{b}{a} + \sqrt{\dfrac{b^2}{a^2} - 1}\right) + 2\pi b^2.
\end{aligned}
$$

4) It follows from the figure that the meridian curve is a half circle of radius $\dfrac{a}{2}$. Thus the integral $\int_{\mathcal{O}} dS$ is equal to the surface area of the sphere, i.e.

$$
\int_{\mathcal{O}} dS = 4\pi \left(\frac{a}{2}\right)^2 = \pi a^2
$$

according to **Example 2.2.3** with $a = b$.

ALTERNATIVELY,

$$
\varrho = \sqrt{\left(\frac{a}{2}\right)^2 - \left(z - \frac{a}{2}\right)^2}, \qquad \text{for } z \in [0.a],
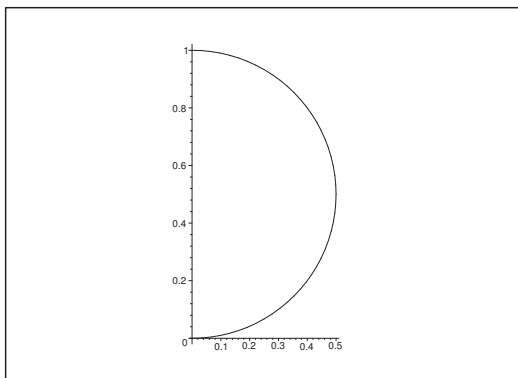$$

Figure 41: The meridian curve $\mathcal{M}$ of **Example 2.2.4** and **Example 1.3.8** for $a = 1$.

in rectangular coordinates, so

$$ds = \sqrt{1 + \frac{\left(z - \dfrac{a}{2}\right)^2}{\left(\dfrac{a}{2}\right)^2 - \left(z - \dfrac{a}{2}\right)^2}}\, dz = \frac{a}{2}\,\frac{1}{\sqrt{\left(\dfrac{a}{2}\right)^2 - \left(z - \dfrac{a}{2}\right)^2}}\, dz.$$

Hence

$$\int_{\mathcal{O}} dS = 2\pi \int_0^a \sqrt{\left(\frac{a}{2}\right)^2} \cdot \frac{a}{2} \cdot \frac{1}{\sqrt{\left(\dfrac{a}{2}\right)^2 - \left(z - \dfrac{a}{2}\right)^2}}\, dz = 2\pi \cdot \frac{a}{2} \cdot a = \pi a^2.$$

ALTERNATIVELY, $r = a\,\cos\theta$, $\theta \in \left[0, \dfrac{\pi}{2}\right]$, in spherical coordinates, and $\varrho = r\,\sin\theta = a\,\sin\theta\,\cos\theta$, and

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = a\, d\theta,$$

thus

$$\int_{\mathcal{O}} dS = 2\pi \int_0^{\frac{\pi}{2}} a\,\sin\theta\,\cos\theta \cdot a\, d\theta = a^2\pi \left[\sin^2\theta\right]_0^{\frac{\pi}{2}} = a^2\pi.$$

5) Since $ds = \sqrt{1 + 9z^4}\, dz$, we get

$$\begin{aligned}
\int_{\mathcal{O}} dS &= 2\pi \int_0^1 z^3\sqrt{1 + 9z^4}\, dz = \frac{2\pi}{4} \int_0^1 \sqrt{1 + 9t}\, dt \\
&= \frac{\pi}{2} \cdot \frac{1}{9} \cdot \frac{2}{3} \left[(1 + 9t)^{\frac{3}{2}}\right]_0^1 = \frac{\pi}{27}\left(10\sqrt{10} - 1\right).
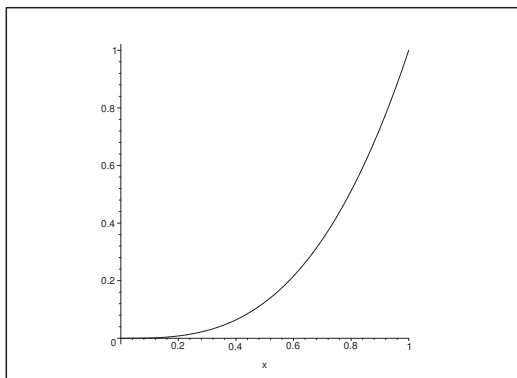\end{aligned}$$

Figure 42: The meridian curve $\mathcal{M}$ of **Example 2.2.5**.

Example 2.3 *Consider the space curve $\mathcal{K}$ given by the parametric description*

$$\mathbf{r}(t) = \left(3\cos t - 2\cos^3 t, 2\sin^3 t, 3\cos t\right), \qquad t \in \left[0, \frac{\pi}{2}\right].$$

1. *Show that the curve has a tangent at the points of the curve corresponding to $t \in \left]0, \frac{\pi}{2}\right]$.*

2. *Show that the curve has a tangent at the point corresponding to $t = 0$.*

3. *Find the length of $\mathcal{K}$.*

The curve $\mathcal{K}$ is projected onto the (X, Y)-plane in a curve $\mathcal{K}^*$. Let $\mathcal{O}$ denote the surface of revolution which is obtained by rotating the curve $\mathcal{K}^*$ once around the X-axis; and $\mathcal{C}$ denotes the cylinder surface which has $\mathcal{K}^*$ as its leading curve and the Z-axis as its direction of generators, and which is lying between the curve $\mathcal{K}$ and the plane $z = -x$.

4. *Find the area of $\mathcal{O}$.*

5. *Find the area of $\mathcal{C}$.*

A Length of a space curve; area of a surface of revolution and a cylinder surface.

D Calculate $\mathbf{r}'(t)$ and show that $\mathbf{r}'(t) \neq \mathbf{0}$ in $\left]0, \frac{\pi}{2}\right[$. Check what happens for $t \to 0$. Find $\|\mathbf{r}'(t)\|$. Finally, compute the surface areas.

I 1) We get by a differentiation

$$\begin{aligned}
\mathbf{r}'(t) &= \left(-3\sin t + 6\cos^2 t \, \sin t, 6\sin^2 t \, \cos t, -3\sin t\right) \\
&= 3\sin t \left(2\cos^2 t - 1, 2\sin t \, \cos t, -1\right) \\
&= 3\sin t \left(cos 2t, \sin 2t, -1\right).
\end{aligned}$$

Clearly, $\mathbf{r}'(t) \neq \mathbf{0}$ for $t \in \left]0, \frac{\pi}{2}\right[$, hence the curve has a tangent in each of the points corresponding to $t \in \left]0, \frac{\pi}{2}\right[$.

2) It follows from

$$\frac{1}{3\sin t}\mathbf{r}'(t) = (\cos 2t, \sin 2t, -1) \to (1, 0, -1) \neq (0, 0, 0) \qquad \text{for } t \to 0,$$

that the curve has a tangent (actually a "half tangent") at the point corresponding to $t = 0$.
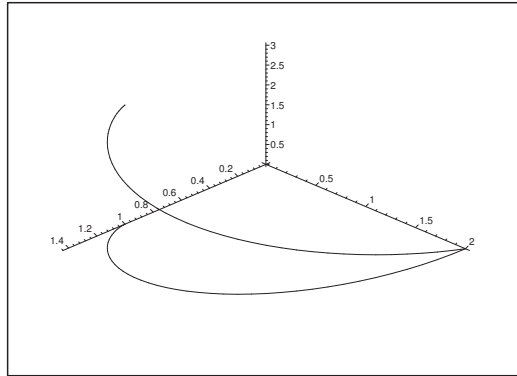
Figure 43: The space curve $\mathcal{K}$ and its projection onto the (X, Y)-plane.

3) From

$$\|\mathbf{r}'(t)\|^2 = (3\sin t)^2 \cdot \left\{\cos^2 2t + \sin^2 2t + 1\right\} = (3\sqrt{2}\sin t)^2,$$

follows that the length of the curve $\mathcal{K}$ is

$$\ell = 3\sqrt{2} \int_0^{\frac{\pi}{2}} \sin t \, dt = 3\sqrt{2}[-\cos t]_0^{\frac{\pi}{2}} = 3\sqrt{2}.$$

The projection of the curve onto the (X, Y)-plane has the parametric description

$$\tilde{\mathbf{r}}'(t) = \left(\cos t\{3 - 2\cos^2 t\}, 2\sin^3 t, 0\right), \qquad t \in \left[0, \frac{\pi}{2}\right].$$

By glancing at 1) we get

$$\tilde{\mathbf{r}}'(t) = 3\sin t \, (\cos 2t, \sin 2t, 0) \quad \text{og} \quad \|\tilde{\mathbf{r}}'(t)\| = 3\sin t.$$

4) The surface area of $\mathcal{O}$ is

$$
\begin{aligned}
\text{area}(\mathcal{O}) &= \int_0^{\frac{\pi}{2}} 2\pi \tilde{y}(t) \cdot \|\tilde{\mathbf{r}}'(t)\| \, dt \\
&= 2\pi \int_0^{\frac{\pi}{2}} 2\sin^3 t \cdot 3\sin t \, dt = 3\pi \int_0^{\frac{\pi}{2}} \left(2\sin^2 t\right)^2 dt \\
&= 3\pi \int_0^{\frac{\pi}{2}} (1 - \cos 2t)^2 dt = 3\pi \int_0^{\frac{\pi}{2}} (1 - 2\cos 2t + \cos^2 2t) \, dt \\
&= \frac{3\pi^2}{2} - 3\pi[\sin 2t]_0^{\frac{\pi}{2}} + \frac{3\pi}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 4t) \, dt = \frac{3\pi^2}{2} + \frac{3\pi^2}{4} = \frac{9\pi^2}{4}.
\end{aligned}
$$

5) The surface area of $\mathcal{C}$ is

$$
\begin{aligned}
\text{area}(\mathcal{C}) &= \int_0^{\frac{\pi}{2}} \{3\cos t + x(t)\} \cdot \|\tilde{\mathbf{r}}'(t)\| \, dt = \int_0^{\frac{\pi}{2}} \{6\cos t - 2\cos^3 t\} \cdot 3\sin t \, dt \\
&= 3 \int_0^{\frac{\pi}{2}} \{3 - \cos^2 t\} \cdot \sin 2t \, dt = \frac{3}{2} \int_0^{\frac{\pi}{2}} (5 - \cos 2t) \cdot \sin 2t \, dt \\
&= \frac{15}{2} \int_0^{\frac{\pi}{2}} \sin 2t \, dt - \frac{3}{4} \int_0^{\frac{\pi}{2}} \sin 4t \, dt = \frac{15}{2} \left[-\frac{1}{2}\cos 2t\right]_0^{\frac{\pi}{2}} + \frac{3}{16}[\cos 4t]_0^{\frac{\pi}{2}} = \frac{15}{2}.
\end{aligned}
$$

Example 2.4 .

1. *Find the length of the curve $\mathcal{K}$ given by the parametric description*

$$\mathbf{r}(t) = \left(3\left(1 - t^2\right)^2, 8t^3, 0\right), \qquad t \in [0, 1].$$

Choose $\mathcal{K}$ as the leading curve for a cylinder surface $\mathcal{C}$ with the Z-axis as its direction of the generators.

2. *Find the area of that part of $\mathcal{C}$, which lies between the curve $\mathcal{K}$ and the plane of equation $z = 1 + y$.*

A Curve length; surface area.

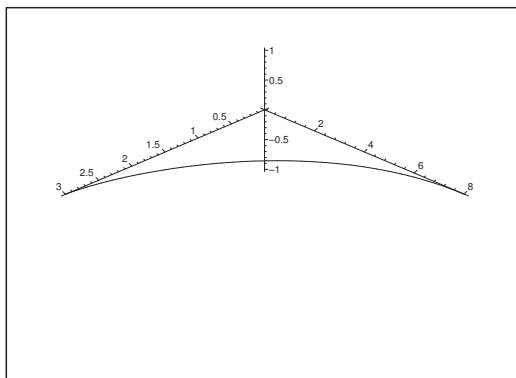**D** Find $\|\mathbf{r}'(t)\|$ and integrate. Then find the surface area.

Figure 44: The space curve $\mathcal{K}$.

I 1) We get from
$$\mathbf{r}'(t) = \left(-12t\left(1 - t^2\right), 24t^2, 0\right) = 12t\left(t^2 - 1, 2t, 0\right)$$
that
$$\|\mathbf{r}'(t)\|^2 = (12t)^2 \cdot \left\{t^4 - 2t^2 + 1 + 4t^2\right\} = (12t)^2 \left(t^2 + 1\right)^2,$$
and thus
$$\|\mathbf{r}'(t)\| = 12t\left(t^2 + 1\right).$$
Hence, the arc length is
$$\ell = \int_0^1 \|\mathbf{r}'(t)\| \, dt = \int_0^1 12t\left(t^2 + 1\right) \, dt = 6\int_{u=t^2=0}^1 (u + 1)\, du = \left[3u^2 + 6u\right]_0^1 = 9.$$

2) The surface area is
$$\begin{aligned}
A &= \int_0^1 [1 + y]_{y=8t^3} \cdot \|\mathbf{r}'(t)\| \, dt = \int_0^1 \left(1 + 8t^3\right) \cdot 12t\left(t^2 + 1\right)\, dt \\
&= \ell + 96\int_0^1 \left(t^6 + t^4\right)\, dt = 9 + 96\left(\frac{1}{7} + \frac{1}{5}\right) = \frac{1467}{35}.
\end{aligned}$$

Example 2.5 *Find the area of that part $\mathcal{C}$ of the cylinder surface of equation $x^2 + y^2 = 9$, which is bounded by the plane $z = 0$ and the surface of equation $z = 1 + x^2$.*

A Area of a part of a cylinder surface.

D Just compute.

I When we integrate along the curve
$$\mathcal{K}: \quad (x, y) = (3\cos\varphi, 3\sin\varphi),$$
we get
$$\text{area}(\mathcal{C}) = \int_{\mathcal{K}} (1 + x^2) \, ds = \int_0^{2\pi} \left(1 + 9\cos^2\varphi\right) \cdot 3 \, d\varphi = 6\pi + 27\pi = 33\pi.$$

Example 2.6 *Given a curve $\mathcal{K}$ in the (X, Z)-plane by*

$$z = \left(x - \frac{4}{9} \right)^{\frac{3}{2}}, \qquad x \in [1, 2].$$

1) Find the length of $\mathcal{K}$.

2) Find the area of that surface $\mathcal{F}$, which is created when $\mathcal{K}$ is rotated once around the Z-axis.

A Curve length, surface area.

D Find the line element

$$ds = \sqrt{1 + \left(\frac{dz}{dx} \right)} \, dx$$

and compute $\int_{\mathcal{K}} ds$ and $2\pi \int_{\mathcal{K}} x \, ds$.
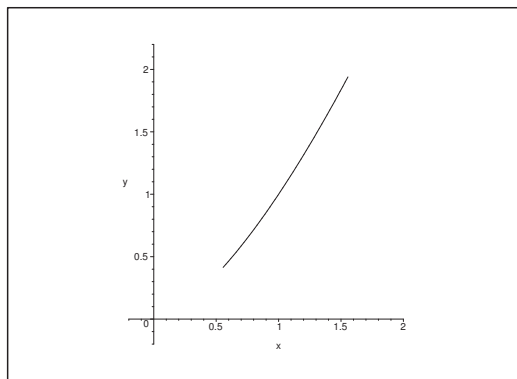

Figure 45: The curve $\mathcal{K}$.

I 1) We get from

$$\frac{dz}{dx} = \frac{3}{2} \sqrt{x - \frac{4}{9}},$$

the line element

$$ds = \sqrt{1 + \frac{9}{4} \left(x - \frac{4}{9} \right)} \, dx = \frac{3}{2} \sqrt{x} \, dx,$$

and the curve length becomes

$$\ell = \frac{3}{2} \int_1^2 \sqrt{x} \, dx = [x\sqrt{x}]_1^2 = 2\sqrt{2} - 1.$$

2) The surface area is according to a formula

$$\text{area}(\mathcal{F}) = 2\pi \int_{\mathcal{K}} x \, ds = 2\pi \cdot \frac{3}{2} \int_1^2 x\sqrt{x} \, dx = 2\pi \cdot \frac{3}{2} \cdot \frac{2}{5} \left[x^2 \sqrt{x} \right]_1^2 = \frac{6\pi}{5} \left(4\sqrt{2} - 1 \right).$$
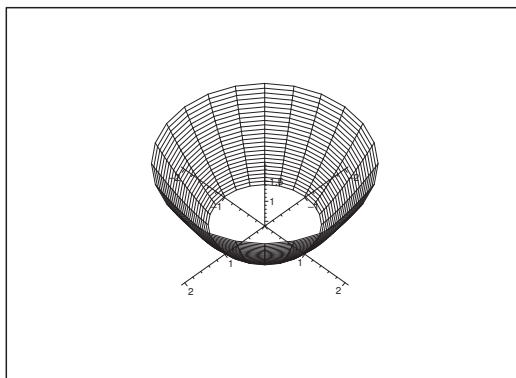
Figure 46: The surface of revolution $\mathcal{F}$.

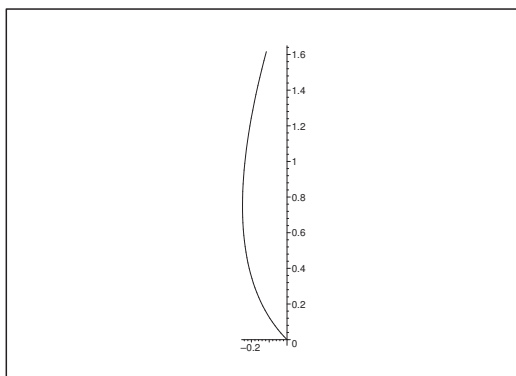**Example 2.7** *A cylinder surface $\mathcal{C}$ has its generators parallel to the Z-axis and its leading curve $\mathcal{K}$ in the (X, Y)-plane is given by the parametric description*

$$\mathbf{r}(t) = \left(t^2 - t, t^2 + t\right), \qquad t \in \left[0, \frac{\sqrt{3}}{2}\right].$$

Find the area of that part $\mathcal{F}$ of $\mathcal{C}$, which is bounded by the plane $z = 0$ and the plane $z = 8y - 8x$.
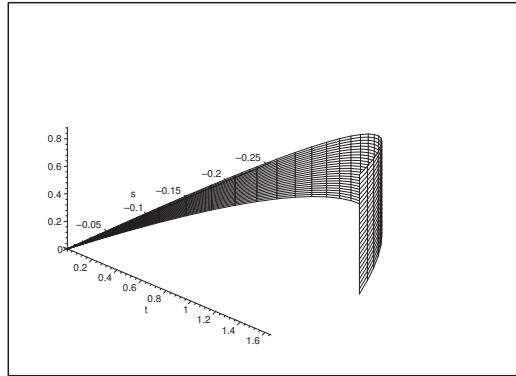
A Surface area.

D First find $\mathbf{r}'(t)$.

Figure 47: The curve $\mathcal{K}$.

I First note that $z = 8y - 8x = 16t \geq 0$ on $\mathcal{K}$. Then

$$\mathbf{r}'(t) = (2t - 1, 2t + 1), \qquad \|\mathbf{r}'(t)\| = \sqrt{2} \cdot \sqrt{4t^2 + 1}.$$

Figure 48: The surface $\mathcal{F}$.

When we insert into the formula of the area of a cylinder surface with a leading curve, then

$$
\begin{aligned}
\text{area}(\mathcal{F}) &= \int_{\mathcal{K}} (8y - 8x)\, ds = \sqrt{2} \int_0^{\frac{\sqrt{3}}{2}} 16t \cdot \sqrt{4t^2 + 1}\, dt = 2\sqrt{2} \left[\frac{2}{3} \left(\sqrt{4t^2 + 1} \right)^3 \right]_0^{\frac{\sqrt{3}}{2}} \\
&= \frac{4\sqrt{2}}{3} \left\{ \left(\sqrt{4 \cdot \frac{3}{4} + 1} \right)^3 - 1 \right\} = \frac{28\sqrt{2}}{3}.
\end{aligned}
$$

Example 2.8 *Find an equation of the tangent plane of the graph of the function*

$$g(x, y) = \sqrt{2xy}, \qquad (x, y) \in [1, 4] \times [1, 4]$$

at the point $(x, y) = (2, 2)$. *Find the area of the graph.*

A Tangent plane and surface area.

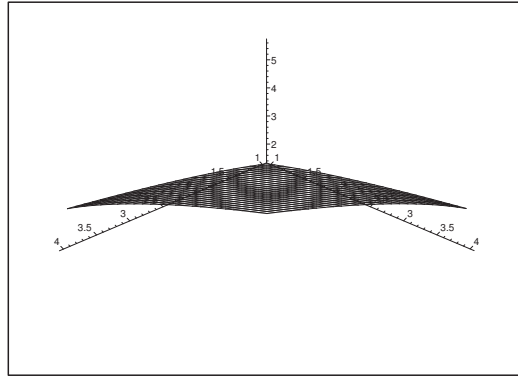**D** Find the approximating polynomial of at most first degree at the point of contact.

Figure 49: The graph of f.

An equation of the tangent plane of $z = g(x, y)$ is

$$
\begin{aligned}
z &= P_1(x, y) = g(2, 2) + \nabla g(2, 2) \cdot (x - 2, y - 2) \\
&= 2\sqrt{2} + \left(\sqrt{\frac{y}{2x}}, \sqrt{\frac{x}{2y}} \right)_{(x,y)=(2,2)} \cdot (x - 2, y - 2) \\
&= 2\sqrt{2} + \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \cdot (x - 2, y - 2) = 2\sqrt{2} + \frac{1}{\sqrt{2}} (x + y - 4) = \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y,
\end{aligned}
$$

thus

$$x + y - \sqrt{2}\, z = 0.$$

Then according to some formula, the area of the graph is

$$
\begin{aligned}
\int_E \sqrt{1 + \| \nabla g \|^2}\, dx\, dy &= \int_1^4 \left\{ \int_1^4 \sqrt{1 + \frac{y}{2x} + \frac{x}{2y}}\, dx \right\} dy \\
&= \int_1^4 \left\{ \int_1^4 \sqrt{\frac{1}{2xy} (2xy + y^2 + x^2)}\, dx \right\} dy = \int_1^4 \left\{ \int_1^4 \sqrt{\frac{(x + y)^2}{2xy}}\, dx \right\} dy \\
&= \int_1^4 \frac{1}{\sqrt{2y}} \left\{ \int_1^4 \left(x^{\frac{1}{2}} + y\, x^{-\frac{1}{2}} \right) dx \right\} dy = \frac{1}{\sqrt{2}} \int_1^4 \frac{1}{\sqrt{y}} \left[\frac{2}{3} x^{\frac{3}{2}} + 2y\, x^{\frac{1}{2}} \right]_{x=1}^4 dy \\
&= \frac{1}{\sqrt{2}} \int_1^4 \frac{1}{\sqrt{y}} \left\{ \frac{2}{3} (8 - 1) + 2y(2 - 1) \right\} dy = \frac{1}{\sqrt{2}} \int_1^4 \left\{ \frac{14}{3} y^{-\frac{1}{2}} + 2y^{\frac{1}{2}} \right\} dy \\
&= \frac{1}{\sqrt{2}} \left[\frac{28}{3} y^{\frac{1}{2}} + \frac{4}{3} y^{\frac{3}{2}} \right]_1^4 = \frac{4}{3\sqrt{2}} \{ 7(2 - 1) + (8 - 1) \} = \frac{28\sqrt{2}}{3}.
\end{aligned}
$$

3 Improper surface integrals

Example 3.1 *Given the meridian curve $\mathcal{M}$ of the parametric description*

$$\varrho = a \cos t, \quad z = a\{\ln(1 + \sin t) - \ln \cos t - \sin t\}, \qquad t \in \left[0, \frac{\pi}{2}\right[$$

When this is rotated we obtain a surface of revolution $\mathcal{O}$ (half of the pseudo-sphere), which stretches into infinity along the positive part of the Z-axis.
Find the integral which gives the area of that part of $\mathcal{O}$, which corresponds to $[0, T]$, where $T < \dfrac{\pi}{2}$.
Then find the area of the pseudo-sphere by letting $T \to \dfrac{\pi}{2}$.

A Surface area of an infinite surface of revolution; improper surface integral.

D First find the curve element ds on $\mathcal{M}$. Then compute the surface area of $\mathcal{O}_T$, i.e. the surface corresponding to $t \in [0, T]$, where $T < \dfrac{\pi}{2}$. This means that we shall compute

$$2\pi \int_{\mathcal{O}_T} \varrho(t) \, ds.$$

Finally, take the limit $T \to \dfrac{\pi}{2}-$.

I First calculate

$$
\begin{aligned}
\mathbf{r}'(t) &= a\left(-\sin t, \frac{\cos t}{1 + \sin t} + \frac{\sin t}{\cos t} - \cos t\right) = a\left(-\sin t, \frac{\cos t \cdot (1 - \sin t)}{1 - \sin^2 t} + \frac{\sin t}{\cos t} - \cos t\right) \\
&= a\left(-\sin t, \frac{1 - \sin t + \sin t - \cos^2 t}{\cos t}\right) = a \sin t \cdot (-1, \tan t).
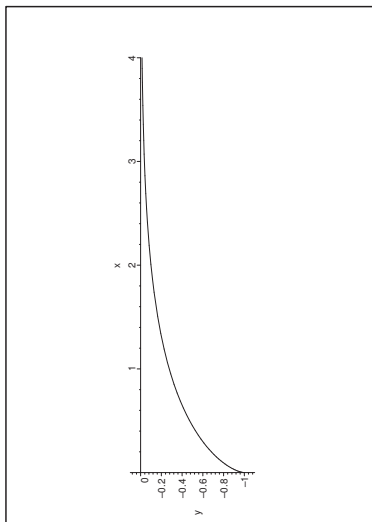\end{aligned}
$$

Figure 50: The meridian curve of the pseudo-sphere.

Hence

$$ds = \|\mathbf{r}'(t)\|\, dt = a|\sin t|\sqrt{1 + \tan^2 t}\, dt = a \left| \frac{\sin t}{\cos t} \right| dt = a \tan t\, dt,$$

and accordingly,

$$\begin{aligned}
\text{area}(\mathcal{O}_T) &= 2\pi \int_{\mathcal{O}_T} \varrho(t)\, ds = 2\pi \int_0^T a\cos t \cdot a\tan t\, dt \\
&= 2\pi a^2 \int_0^T \sin t\, dt = 2\pi a^2 (1 - \cos T).
\end{aligned}$$

Finally, by taking the limit we find the improper surface area

$$\text{areal}(\mathcal{O}) = \lim_{T \to \frac{\pi}{2}-} \text{areal}(\mathcal{O}_T) = 2\pi a^2.$$

REMARK. Notice that the "half" pseudo-sphere has the same surface area as the usual upper half sphere of radius a. ◊

Example 3.2 *Check in each of the following cases if the given surface integral is convergent or divergent; in case of convergency, find the value.*

1) *The surface integral $\int_{\mathcal{F}} \dfrac{1}{(a+4z)^2}\, dS$ over the surface $\mathcal{F}$ given by $az = x^2 + y^2$, $(x,y) \in \mathbb{R}^2$.*

2) *The surface integral $\int_{\mathcal{F}} \dfrac{x^2}{z^2 + a^2}\, dS$ over the surface $\mathcal{F}$ given by $x^2 + y^2 = a^2$, $z \in \mathbb{R}$.*

3) *The surface integral $\int_{\mathcal{F}} y^2 \exp\left(-\dfrac{|z|}{a}\right) dS$ over the surface $\mathcal{F}$ given by $x^2 + y^2 = a^2$, $z \in \mathbb{R}$.*

4) *The surface integral $\int_{\mathcal{F}} \dfrac{1}{z(x+y)}\, dS$ over the surface $\mathcal{F}$ given by $z = \sqrt{2xy}$, $(x,y) \in [a, +\infty[^2$.*

A Improper surface integral.

D First analyze why the integral is improper. Then truncate the surface and split it into the positive and the negative part of the integrand. Finally take the limit.

I 1) The surface is a paraboloid of revolution.

$$z = \frac{1}{a}(x^2 + y^2) = \frac{1}{a}\varrho \geq 0.$$

The integrand is $\geq \dfrac{1}{a^2} > 0$ everywhere on the surface.

The surface is described as the graph of the equation $z = \dfrac{1}{a}(x^2 + y^2)$, so the weight function becomes

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \left(\frac{2x}{a}\right)^2 + \left(\frac{2y}{a}\right)^2} = \sqrt{1 + \frac{4}{a^2}\varrho^2}.$$

We choose the truncated domain in polar coordinates as $0 \le \varrho \le R$. It follows from the above that the area element is

$$dS = \sqrt{1 + \frac{4}{a^2}\,\varrho^2}\,\varrho\,d\varrho\,d\varphi,$$

hence the surface integral over the truncated surface $\mathcal{F}_R$ is

$$
\begin{aligned}
\int_{\mathcal{F}_R} \frac{1}{(a+4z)^2}\,dS &= \int_0^{2\pi}\left\{\int_0^R \frac{1}{\left(a+\dfrac{4\varrho^2}{a}\right)^2}\sqrt{1+\frac{4}{a^2}}\,\varrho\,d\varrho\right\}d\varphi \\[2mm]
&= \frac{2\pi}{a^2}\int_0^R \left\{1+\frac{4}{a^2}\,\varrho^2\right\}^{-\frac{3}{2}}\cdot\frac{1}{2}\cdot\frac{a^2}{4}\cdot\frac{4}{a^2}\cdot 2\varrho\,d\varrho \\[2mm]
&= \frac{2\pi}{a^2}\cdot\frac{a^2}{8}\left[-\frac{2}{\sqrt{1+\dfrac{4}{a^2}\,\varrho^2}}\right]_0^R = \frac{\pi}{2}\left(1-\frac{1}{\sqrt{1+\dfrac{4}{a^2}\,R^2}}\right).
\end{aligned}
$$

This expression clearly converges for $R \to +\infty$, thus the improper surface integral is convergent of the value

$$\int_{\mathcal{F}} \frac{1}{(a+4z)^2} \, dS = \lim_{R \to +\infty} \frac{\pi}{2} \left(1 - \frac{1}{\sqrt{1 + \frac{4}{a^2} R^2}} \right) = \frac{\pi}{2}.$$

2) The surface is an infinite cylinder surface with the circle in the XY-plane of centrum $(0,0)$ and radius a as its leading curve. By using semi polar coordinates we get

$$x = a \cos \varphi, \quad y = a \sin \varphi, \quad z = z, \qquad \varphi \in [0, 2\pi], \quad z \in \mathbb{R},$$

and

$$dS = a \, d\varphi \, dz.$$

The integrand is positive, so we choose the truncation $|z| \le A$. Then

$$\begin{aligned}
\int_{\mathcal{F}_A} \frac{x^2}{z^2 + a^2} \, dS &= \int_0^{2\pi} \left\{ \int_{-A}^A \frac{a^2 \cos^2 \varphi}{z^2 + a^2} \cdot a \, dz \right\} d\varphi \\
&= \int_0^{2\pi} \cos^2 \varphi \, d\varphi \cdot a^2 \int_{-A}^A \frac{1}{1 + \left(\frac{z}{a}\right)^2} \cdot \frac{1}{a} \, dz = a^2 \pi \cdot 2 \mathrm{Arctan} \left(\frac{A}{a} \right).
\end{aligned}$$

This expression converges for $A \to +\infty$, and we conclude that the improper surface integral is convergent with the value

$$\int_{\mathcal{F}} \frac{x^2}{z^2 + a^2} \, dS = \lim_{A \to +\infty} a^2 \pi \cdot 2 \mathrm{Arctan} \left(\frac{A}{a} \right) = a^2 \pi^2.$$

3) By using semi polar coordinates it is seen that

$$x = a \cos \varphi, \quad y = a \sin \varphi, \quad z = z, \qquad \varphi \in [0, 2\pi], \quad z \in \mathbb{R}.$$

The surface element is

$$dS = a \, d\varphi \, dz.$$

The integrand is positive everywhere, so we choose the truncation $|z| \le A$. Then

$$\begin{aligned}
\int_{\mathcal{F}_A} y^2 \exp \left(-\frac{|z|}{a} \right) dS &= \int_0^{2\pi} \left\{ \int_{-A}^A a^2 \sin^2 \varphi \cdot \exp \left(-\frac{|z|}{a} \right) a \, dz \right\} d\varphi \\
&= a^3 \cdot \pi \cdot 2 \int_0^A \exp \left(-\frac{z}{a} \right) dz = 2a^4 \pi \cdot \left[-\exp \left(-\frac{z}{a} \right) \right]_0^A \\
&= 2\pi a^4 \left\{ 1 - \exp \left(-\frac{A}{a} \right) \right\}.
\end{aligned}$$

This expression is clearly convergent for $A \to +\infty$, thus the improper surface integral is convergent with the value

$$\int_{\mathcal{F}} y^2 \exp \left(-\frac{|z|}{a} \right) dS = \lim_{A \to +\infty} \int_{\mathcal{F}_A} y^2 \exp \left(-\frac{|z|}{a} \right) dS = 2\pi a^4.$$

4) When the surface is the graph of $z = \sqrt{2xy}$ for $x, y \geq a$, then the surface element is

$$
\begin{aligned}
dS &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = \sqrt{1 + \frac{y}{2x} + \frac{x}{2y}}\, dx\, dy \\
&= \sqrt{\frac{2xy + y^2 + x^2}{2xy}}\, dx\, dy = \frac{x+y}{\sqrt{2xy}}\, dx\, dy.
\end{aligned}
$$

The integrand is given on the surface $\mathcal{F}$ by

$$
\frac{1}{z(x+y)} = \frac{1}{\sqrt{2xy}\,(x+y)}
$$

which is clearly positive, because $x, y \geq a$.

For every $A > a$ we define the truncation by $a \leq x, y \leq A$. Then the surface integral over the corresponding truncated surface $\mathcal{F}_A$ is

$$
\begin{aligned}
\int_{\mathcal{F}_A} \frac{1}{z(x+y)}\, dS &= \int_a^A \left\{ \int_a^A \frac{1}{\sqrt{2xy}\,(x+y)} \cdot \frac{x+y}{\sqrt{2xy}}\, dx \right\} dy \\
&= \frac{1}{2}\left\{\int_a^A \frac{dx}{x}\right\} \cdot \left\{\int_a^A \frac{dy}{y}\right\} = \frac{1}{2}\left\{[\ln t]_a^A\right\}^2 = \frac{1}{2}\left\{\ln\left(\frac{A}{a}\right)\right\}^2 \to +\infty \quad \text{for } A \to +\infty.
\end{aligned}
$$

We conclude that the improper surface integral is divergent.

5) The surface is the same as in **Example 3.2.4**, so the surface element is

$$
dS = \frac{x+y}{\sqrt{2xy}}\, dx\, dy.
$$

The integrand is on the surface $\mathcal{F}$ given by

$$
\frac{1}{z^2 xy} = \frac{1}{2xy \cdot xy} = \frac{1}{2} \cdot \frac{1}{x^2} \cdot \frac{1}{y^2}.
$$

This is positive, so we shall again use the truncation $a \leq x, y \leq A$. Then

$$
\begin{aligned}
\int_{\mathcal{F}_A} \frac{1}{z^2 xy}\, dS &= \int_a^A \left\{ \int_a^A \frac{1}{2} \cdot \frac{1}{x^2} \cdot \frac{1}{y^2} \cdot \frac{x+y}{\sqrt{2xy}}\, dx \right\} dy \\
&= \frac{1}{2\sqrt{2}} \int_a^A x^{-\frac{3}{2}}\, dx \cdot \int_a^A y^{-\frac{5}{2}}\, dy + \frac{1}{2\sqrt{2}} \int_a^A x^{-\frac{5}{2}}\, dx \cdot \int_a^A y^{-\frac{3}{2}}\, dy \\
&= \frac{1}{\sqrt{2}}\left[-\frac{2}{\sqrt{x}}\right]_a^A \cdot \left[-\frac{2}{3} \cdot \frac{1}{y\sqrt{y}}\right]_a^A = \frac{2\sqrt{2}}{3}\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{A}}\right) \cdot \left(\frac{1}{a\sqrt{a}} - \frac{1}{A\sqrt{A}}\right) \\
&\to \frac{2\sqrt{2}}{3} \cdot \frac{1}{\sqrt{a}} \cdot \frac{1}{a\sqrt{a}} = \frac{2\sqrt{2}}{3a^2} \quad \text{for } A \to +\infty.
\end{aligned}
$$

The improper surface integral converges towards the value

$$
\int_{\mathcal{F}} \frac{1}{z^2 xy}\, dS = \frac{2\sqrt{2}}{3a^2}.
$$

Example 3.3 *Check in each of the following cases if the given surface integral is convergent or divergent; in case of convergency, find its value.*
Let $\mathcal{S}$ denote the sphere of centrum $(0,0,0)$ and radius a, while $\mathcal{F}$ is given by $az = x^2 + y^2$, $x^2 + y^2 \leq a^2$.

1) $\int_{\mathcal{S}} \dfrac{1}{a-z}\, dS$,

2) $\int_{\mathcal{S}} \sqrt{\dfrac{a}{|z|}}\, dS$,

3) $\int_{\mathcal{F}} \dfrac{1}{a-z}\, dS$,

4) $\int_{\mathcal{F}} \sqrt{\dfrac{a}{z}}\, dS$.

A Improper surface integrals.

D Analyze why the integral is improper. Since the integrands are ≥ 0 in all cases, we shall only find some nice truncations of the surface.

I 1) Since $|z| \leq a$ on $\mathcal{S}$, the integrand is $\dfrac{1}{a-z} > 0$ on $\mathcal{S} \setminus \{(0,0,a)\}$. The integrand tends towards $+\infty$, when $(x,y,z) \to (0,0,a)$ on $\mathcal{S}$.
When we use spherical coordinates on $\mathcal{S}$,

$$x = a\cos\varphi \cdot \sin\theta, \quad y = a\sin\varphi \cdot \sin\theta, \quad z = a\cos\theta,$$

for

$$\varphi \in [0, 2\pi], \qquad \theta \in [0, \pi],$$

it is well-known that

$$dS = a^2 \sin\theta\, d\varphi\, d\theta.$$

The singular point $(0,0,a)$ corresponds to $\theta = 0$, hence we choose the truncation $\theta \in [\varepsilon, \pi]$, where $\varepsilon > 0$ corresponds to the subsurface $\mathcal{S}_\varepsilon$. When we integrate over $\mathcal{S}_\varepsilon$ we get

$$
\begin{aligned}
\int_{\mathcal{S}_\varepsilon} \frac{1}{a-z}\, dS &= \int_0^{2\pi} \left\{ \int_\varepsilon^\pi \frac{1}{a - a\cos\theta}\, a^2 \sin\theta\, d\theta \right\} d\varphi \\
&= 2\pi a \int_\varepsilon^\pi \frac{\sin\theta}{1 - \cos\theta}\, d\theta = 2\pi a\, [\ln(1 - \cos\theta)]_\varepsilon^\pi \\
&= 2\pi a \{\ln 2 - \ln(1 - \cos\varepsilon)\} = 2\pi a \ln \frac{2}{2\sin^2 \frac{\varepsilon}{2}} \\
&= 4a\pi \ln \frac{1}{\sin \dfrac{\varepsilon}{2}} \to +\infty \qquad \text{for } \varepsilon \to 0+,
\end{aligned}
$$

and the improper surface integral is divergent.

2) In this case the integrand is > 0 on $\mathcal{S}_0$, where $\mathcal{S}_0$ is the set of points on $\mathcal{S}$, which is not contained in the XY-plane, where the integrand is not defined. We use again spherical coordinates. Due to the symmetry it suffices to consider the domain

$$\mathcal{S}_\varepsilon : \qquad \varphi \in [0, 2\pi] \quad \text{og} \quad \theta \in \left[0, \frac{\pi}{2} - \varepsilon\right].$$

From $z = a \cos\theta$ and $dS = a^2 \sin\theta\, d\theta\, d\varphi$, it follows by insertion that

$$\int_{\mathcal{S}_\varepsilon} \sqrt{\frac{a}{|z|}}\, dS = \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}-\varepsilon} \sqrt{\frac{a}{a\,\cos\theta}} \cdot a^2 \sin\theta\, d\theta \right\} d\varphi$$

$$= 2\pi a^2 \int_0^{\frac{\pi}{2}-\varepsilon} \frac{\sin\theta}{\sqrt{\cos\theta}}\, d\theta = 2\pi a^2 \left[-2\sqrt{\cos\theta} \right]_0^{\frac{\pi}{2}-\varepsilon}$$

$$= 4\pi a^2 \{ 1 - \sqrt{\sin\varepsilon} \} \to 4\pi a^2 \qquad \text{for } \varepsilon \to 0+ .$$

We conclude that the improper surface integral is convergent.
For symmetric reasons the value is

$$\int_{\mathcal{S}} \sqrt{\frac{a}{|z|}}\, dS = \lim_{\varepsilon \to 0+} 2 \int_{\mathcal{S}_\varepsilon} \sqrt{\frac{a}{|z|}}\, dS = 2 \cdot 4\pi a^2 = 8\pi a^2 .$$

3) The surface is the graph of $z = \dfrac{1}{a}(x^2 + y^2) = \dfrac{\varrho^2}{a}$, so the area element is

$$
\begin{aligned}
dS &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = \sqrt{1 + \left(\frac{2x}{a}\right)^2 + \left(\frac{2y}{a}\right)^2}\, dx\, dy \\
&= \frac{1}{a}\sqrt{a^2 + 4(x^2 + y^2)}\, dx\, dy = \frac{1}{a}\sqrt{a^2 + 4\varrho^2}\cdot \varrho\, d\varrho\, d\varphi.
\end{aligned}
$$

The integrand is the same as in **Example 3.3.1**, and since $z \le a$ on $\mathcal{F}$, it is positive for $z < a$.
We choose the truncation in polar coordinates by

$$\mathcal{F}_\varepsilon: \qquad 0 \le \varrho \le a - \varepsilon, \quad \varphi \in [0, 2\pi].$$

Then by insertion,

$$
\begin{aligned}
\int_{\mathcal{F}_\varepsilon} \frac{1}{a - z}\, dS &= \int_0^{2\pi} \left\{ \int_0^{a-\varepsilon} \frac{1}{a - \dfrac{\varrho^2}{a}} \cdot \frac{1}{a}\sqrt{a^2 + 4\varrho^2}\, \varrho\, d\varrho \right\} d\varphi \\
&= 2\pi \int_0^{a-\varepsilon} \frac{\varrho}{a^2 - \varrho^2}\sqrt{a^2 + 4\varrho^2}\, d\varrho \ge a\pi \int_0^{a-\varepsilon} \frac{1}{a^2 - \varrho^2}\cdot 2\varrho\, d\varrho \\
&= a\pi \left[-\ln\left(a^2 - \varrho^2\right) \right]_0^{a-\varepsilon} \\
&= a\pi \left\{ \ln a^2 - \ln\left(a^2 - (a - \varepsilon)^2\right) \right\} \to +\infty
\end{aligned}
$$

for $\varepsilon \to 0+$, and the improper surface integral is divergent.

4) The singular point is $(0, 0, 0)$. We choose the truncation

$$\mathcal{F}_\varepsilon: \qquad \varphi \in [0, 2\pi], \quad \varrho \in [\varepsilon, a],$$

and

$$z = \frac{\varrho^2}{a} > 0, \qquad dS = \frac{1}{a}\sqrt{a^2 + 4\varrho^2}\, \varrho\, d\varrho\, d\varphi.$$

Then by insertion

$$
\begin{aligned}
\int_{\mathcal{F}_\varepsilon} \sqrt{\frac{a}{z}}\, dS &= \int_0^{2\pi} \left\{ \int_\varepsilon^a \sqrt{\frac{a^2}{\varrho^2}} \cdot \frac{1}{a}\sqrt{a^2 + 4\varrho^2}\, \varrho\, d\varrho \right\} d\varphi \\
&= 2\pi a \int_\varepsilon^a \sqrt{1 + \left(\frac{2\varrho}{a}\right)^2}\, d\varrho \qquad\qquad \left[\frac{2\varrho}{a} = \sinh t\right] \\
&= 2\pi a \int_{\varrho = \varepsilon}^a \sqrt{1 + \sinh^2 t}\cdot \frac{a}{2}\cosh t\, dt = \pi a^2 \int_{\mathrm{Arsinh}\,\frac{2\varepsilon}{a}}^{\mathrm{Arsinh}\,2} \cosh^2 t\, dt \\
&= \frac{\pi a^2}{2} \int_{\mathrm{Arsinh}\,\frac{2\varepsilon}{a}}^{\mathrm{Arsinh}\,2} (1 + \cosh 2t)\, dt = \frac{\pi a^2}{2} \left[t + \frac{1}{2}\sinh 2t \right]_{\mathrm{Arsinh}\,\frac{2\varepsilon}{a}}^{\mathrm{Arsinh}\,2} \\
&= \frac{\pi a^2}{2} \left\{ \mathrm{Arsinh}\, 2 - \mathrm{Arsinh}\,\frac{2\varepsilon}{a} \right\} + \frac{\pi a^2}{2} \left[\sinh t \sqrt{1 + \sinh^2 t} \right]_{\mathrm{Arsinh}\,\frac{2\varepsilon}{a}}^{\mathrm{Arsinh}\,2} \\
&\to \frac{\pi a^2}{2}\,\mathrm{Arsinh}\, 2 + \frac{\pi a^2}{2}\{\ln(2 + \sqrt{5}) + 2\sqrt{5}\}
\end{aligned}
$$

for $\varepsilon \to 0+$, and the improper surface integral converges towards the value

$$\int_{\mathcal{F}} \sqrt{\frac{a}{z}}\, dS = \frac{\pi a^2}{2}\{\ln(2+\sqrt{5}) + 2\sqrt{5}\}.$$

Example 3.4 *Check if the surfaces of the bodies of revolution of the* **Examples 2.3** *and* **2.4** *in* Calculus 2c-6, Example of Space Integrals, *can be given a finite area.*
(These shall not be computed).

A Improper surface integrals.

D Consult **Example 2.3** and **2.4** of *Calculus 2c-6, Examples of Space Integrals.* Since we are only dealing with areas, the integrand is automatically positive. Truncate suitably before the computation of the surface integral, and then take the limit.

I **Example 2.3.** The curve $\mathcal{K}$ of the equation

$$y^2(a-x) = x^3$$

is rotated around the asymptote $x = a$.
For symmetric reasons it suffices to consider $y \geq 0$, thus

$$y = x\sqrt{\frac{x}{a-x}} = x^{\frac{3}{2}}(a-x)^{-\frac{1}{2}}.$$

It was shown in **Example 2.3** that

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{x}{(a-x)^3}} \cdot (3a - 2x).$$

The length of the circle C_x (around the line $x = a$) at the height $y(x)$ is $2\pi(a-x)$, [In fact, $0 \leq x < a$].
If we truncate at the height $y(x_0)$ corresponding to some $x_0 \in [0, a[$, and remember the symmetry around $y = 0$ we get the corresponding surface area,

$$2\int_0^{x_0} \text{length}\,(C_x) \cdot \frac{dy}{dx}\, dx$$

$$= 2\int_0^{x_0} 2\pi(a-x) \cdot \frac{1}{2}\sqrt{\frac{x}{(a-x)^3}} \cdot (3a-2x)\, dx = 2\pi\int_0^{x_0}\sqrt{\frac{x}{a-x}}\{a + 2(a-x)\}\, dx$$

$$= 2\pi\int_0^{x_0}\left\{a\sqrt{\frac{x}{a-x}} + 2\sqrt{x(a-x)}\right\} dx, \qquad 0 < x_0 < a.$$

We conclude that the surface has a finite area. The only problem is the term $\sqrt{\dfrac{x}{a-x}}$ in the integrand, and

$$0 \leq \sqrt{\frac{x}{a-x}} \leq \sqrt{a} \cdot \frac{1}{\sqrt{a-x}} \qquad \text{for } 0 < x < a,$$

and

$$\int_0^{x_0} \frac{dx}{\sqrt{a-x}} = [-2\sqrt{a-x}]_0^{x_0} = 2\{\sqrt{a} - \sqrt{a-x_0}\},$$

which converges towards $2\sqrt{a}$ for $x_0 \to a$. Since the area of the surface is smaller than this value, we conclude that the improper surface integral exists.

REMARK. One can in fact find the exact value. Putting $t = \dfrac{x}{a-x}$ we get

$$x = \frac{at}{t+t} = a - \frac{a}{t+1},$$

thus

$$dx = \frac{a}{(t+1)^2}\,dt,$$

and

$$2\pi \int a\sqrt{\frac{x}{a-x}}\,dx = 2\pi a \int \sqrt{t} \cdot \frac{a}{(t+1)^2}\,dt = 2\pi a^2 \int u \cdot \frac{1}{(u^2+1)^2} \cdot 2u\,du$$

$$= 2\pi a^2 \left\{ -\frac{u}{u^2+1} + \int \frac{1}{u^2+1}\,du \right\} = 2\pi a^2 \left\{ \text{Arctan } u - \frac{u}{u^2+1} \right\}$$

$$= 2\pi a^2 \left\{ \text{Arctan }\sqrt{\frac{x}{a-x}} - \frac{\sqrt{\dfrac{x}{a-x}}}{\dfrac{x}{a-x}+1} \right\} = 2\pi a^2 \left\{ \text{Arctan }\sqrt{\frac{x}{a-x}} - \frac{1}{a}\sqrt{x(a-x)} \right\},$$

hence by taking the limit

$$2\pi \int_0^a a\sqrt{\frac{x}{a-x}}\,dx = 2\pi a^2 \left\{ \frac{\pi}{2} - 0 \right\} = \pi^2 a^2.$$

The latter integral is calculated by noting that $y = \sqrt{x(a-x)}$ for $0 \le x \le a$ describes a half circle of centrum $\dfrac{a}{2}$ and radius $\dfrac{a}{2}$, thus

$$4\pi \int_0^a \sqrt{x(a-x)}\,dx = 4\pi \cdot \frac{1}{2} \cdot \pi\left(\frac{a}{2}\right)^2 = \frac{\pi^2 a^2}{2}.$$

Summarizing, the improper surface area is convergent, and its value is

$$\pi^2 a^2 + \frac{\pi^2 a^2}{2} = \frac{3}{2}\pi^2 a^2. \qquad \Diamond$$

Example 2.4. When the curve $\varrho = \dfrac{a^2}{a^2+z^2}$, $z \in \mathbb{R}$, is rotated around the Z-axis, we get an infinite surface which at the height z is cut into a circle $C(x)$ of radius $\varrho(z)$, thus

$$\text{length}(C(x)) = 2\pi\varrho = \frac{2\pi a^3}{a^2+z^2}.$$

By putting

$$\mathcal{F}_k = \{(x,y,z) \in \mathcal{F} \mid |z| \le ka\}, \qquad k > 0,$$

we get

$$
\begin{aligned}
\text{area}\,(\mathcal{F}_k) \;&=\; 2\int_0^{ka} \text{length}(C(x))\,dz = 4\pi a^3 \int_0^{ka} \frac{1}{a^2+z^2}\,dz \\
&=\; 4\pi a^2 \int_0^{k} \frac{1}{1+t^2}\,dt = 4\pi a^2\,\text{Arctan}\,k \\
&\to\; 4\pi a^2\cdot\frac{\pi}{2} = 2\pi^2 a^2 \quad \text{for } k\to +\infty.
\end{aligned}
$$

The improper surface area exists and its value is

$$
\text{area}(\mathcal{F}) = 2\pi^2 a^2.
$$

Example 3.5 *A surface $\mathcal{F}$ is given by the equation*

$$z = 1 + x^2 - y^2, \qquad (x, y) \in \mathbb{R}^2.$$

1. *Indicate the type of the surface and its vertices.*

2. *Find an equation of the tangent plane of $\mathcal{F}$ through the point $(2, 1, 4)$.*

Let q be a positive number. Let $\mathcal{F}(q)$ denote the subset of $\mathcal{F}$, which is given by

$$z = 1 + x^2 - y^2, \qquad x^2 + y^2 \le q^2.$$

3. *Compute the surface integral*

$$I(q) = \int_{\mathcal{F}(q)} \frac{1}{(z + 3x^2 + 5y^2)^{3/2}} \, dS.$$

4. *Explain shortly that*

$$I = \int_{\mathcal{F}} \frac{1}{(z + 3x^2 + 5y^2)^{3/2}} \, dS$$

is an improper surface integral and prove that I is divergent.

A Surface; tangent plane; surface integral; improper surface integral.

D Identify the type of the surface; e.g. set up a parametric description (or use a formula) and find find the field of the normal vectors. Calculate the surface integral by a reduction theorem. Notice that the integrand is positive, and finally take the limit.
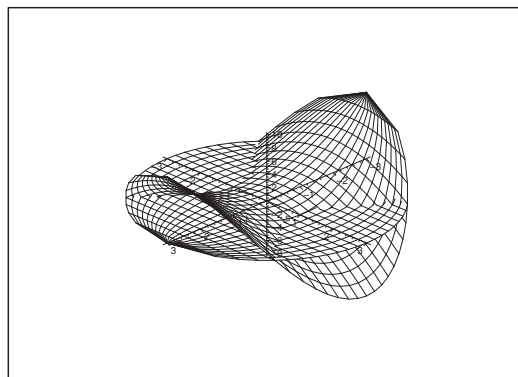

Figure 51: The surface $\mathcal{F}(q)$ for $q = 3$ with the projection $D(q)$ onto the (x, y)-plane.

I 1) It follows from the rearrangement

$$z - 1 = x^2 - y^2$$

that the surface is an equilateral hyperbolic paraboloid with its vertex at $(0, 0, 1)$.

2) It follows from the parametric description

$$\mathbf{r}(x,y) = \left(x, y, 1 + x^2 - y^2\right), \qquad (x,y) \in \mathbb{R}^2,$$

that

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & 2x \\ 0 & 1 & -2y \end{vmatrix} = (-2x, 2y, 1).$$

Then we check if the point $(2,1,4)$ lies on $\mathcal{F}$:

$$1 + x^2 - y^2 = 1 + 4 - 1 = 4 = z,$$

thus $(2,1,4) \in \mathcal{F}$.

The normal vector is in this point

$$(-2x, 2y, 1) = (-4, 2, 1) = \mathbf{N},$$

and an equation of the tangent plane is

$$\begin{aligned} 0 &= \mathbf{N} \cdot (x-2, y-1, z-4) = (-4, 2, 1) \cdot (x-2, y-1, z-4) \\ &= -4x + 2y + z + 8 - 2 - 4 = -4x + 2y + z + 2, \end{aligned}$$

hence by a rearrangement,

$$z = 4x - 2y - 2.$$

3) The parametric domain for $\mathcal{F}(q)$ is the disc in the (x,y)-plane

$$D(q) = \{(x,y) \mid x^2 + y^2 \le q^2\}.$$

Since $z = 1 + x^2 - y^2$ on $\mathcal{F}(q)$, it follows by the theorem of reduction that

$$\begin{aligned} I(q) &= \int_{\mathcal{F}(q)} \frac{1}{(z + 3x^2 + 5y^2)^{3/2}} \, dS = \int_{D(q)} \frac{\|\mathbf{N}(x,y)\|}{(1 + 4x^2 + 4y^2)^{3/2}} \, dx\, dy \\ &= \int_{D(q)} \frac{(1 + 4x^2 + 4y^2)^{1/2}}{(1 + 4x^2 + 4y^2)^{3/2}} \, dx\, dy = \int_{D(q)} \frac{1}{1 + 4(x^2 + y^2)} \, dx\, dy \\ &= \int_0^{2\pi} \left\{ \int_0^q \frac{1}{1 + 4\varrho^2} \, \varrho\, d\varrho \right\} d\varphi = 2\pi \cdot \frac{1}{8} \ln\left(1 + 4q^2\right) = \frac{\pi}{4} \ln\left(1 + 4q^2\right). \end{aligned}$$

4) Now $\mathcal{F}$ is unbounded, so I is an improper surface integral. The integrand is positive on $\mathcal{F}$, hence it suffices to take the limit $q \to +\infty$ for $I(q)$. Then

$$I = \lim_{q \to +\infty} I(q) = \frac{\pi}{4} \lim_{q \to +\infty} \ln\left(1 + 4q^2\right) = +\infty,$$

which proves that the improper surface integral is divergent.